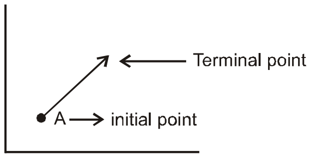
Vectors & Three Dimensional Geometry

What if angry vectors veer Round your sleeping head, and from. There's never need to fear Violence of the poor world's abstract storm. .... Warren, Robert PennNature is an infinite sphere of which the centre is everywhere and the circumference nowhere .... Pascal, Blaise

Vectors and their representation :



Vector quantities are specified by definite magnitude and definite direction. A vector is generally represented by a directed line segment, say $\overrightarrow{AB}$. A is called the **initial point** and B is called the **terminal point**. The magnitude of vector $\overrightarrow{AB}$ is expressed by $|\overrightarrow{AB}|$.

Zero vector :
A vector of zero magnitude i.e. which has the same initial and terminal point, is called a **zero vector.** It is denoted by **O**. **The direction of zero vector is indeterminate.**

Unit vector :
A vector of unit magnitude in the direction of a vector $\vec{a}$ is called unit vector along $\vec{a}$ and is denoted by $\vec{a}$, symbolically $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$.

Equal vectors :

Two vectors are said to be equal if they have the same magnitude, direction and represent the same physical quantity.

Collinear vectors :

Two vectors are said to be collinear if their directed line segments are parallel irrespective of their directions. Collinear vectors are also called **parallel vectors**. If they have the same direction($\rightrightarrows$) they are named as **like vectors** but if they have opposite direction ($\leftrightarrows$) then they are named as **unlike vectors**.

Symbolically, two non-zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if, $\vec{a} = \lambda\vec{b}$, where $\lambda \in R$

$$\vec{a} = \lambda\vec{b} \Leftrightarrow \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right) = \lambda\left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right) \Leftrightarrow a_1 = \lambda b_1,\ a_2 = \lambda b_2,\ a_3 = \lambda b_3 \Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}\ (= \lambda)$$

Vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are collinear if $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3}$

Note : If $\vec{a}, \vec{b}$ are non zero, non–collinear vectors, such that $x\vec{a} + y\vec{b} = x'\vec{a} + y'\vec{b} \Rightarrow x = x'$, $y = y'$, (where x, x', y, y' are scalars)

Example # 1 : Find unit vector of $\hat{i}-2\hat{j}+3\hat{k}$

Solution : $\vec{a} = \hat{i}-2\hat{j}+3\hat{k}$

if $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ then $|\vec{a}| = \sqrt{a_x^2+a_y^2+a_z^2}$

$\therefore$ $|\vec{a}| = \sqrt{14}$ $\Rightarrow$ $\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{\sqrt{14}}\hat{i} - \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}$

Example # 2 : $\vec{a} = (x + 1)\,\hat{i} - (2x + y)\,\hat{j} + 3\hat{k}$ and $\vec{b} = (2x - 1)\,\hat{i} + (2 + 3y)\,\hat{j} + \hat{k}$

find $\vec{x}$ and $\vec{y}$ for which $\vec{a}$ and $\vec{b}$ are parallel.

Solution : $\vec{a}$ and $\vec{b}$ are parallel $\Rightarrow$ $\frac{x+1}{2x-1} = \frac{-(2x+y)}{2+3y} = \frac{3}{1}$ $\Rightarrow$ $x = 4/5$, $y = -19/25$

Coplanar vectors :

A given number vectors are called coplanar if their line segments are all parallel to the same plane. Note that **"two vectors are always coplanar"**.

Multiplication of a vector by a scalar :

If $\vec{a}$ is a vector and m is a scalar, then m is a vector parallel to $\vec{a}$ whose magnitude is $|m|$ times that of $\vec{a}$. This multiplication is called **scalar multiplication**. If $\vec{a}$ and $\vec{a}$ are vectors and m, n are scalars, then :

(i) $m\ (\vec{a}) = (\vec{a})\ m = m\vec{a}$ (ii) $m\ (n\vec{a}) = n(m\vec{a}) = (mn)\vec{a}$

(iii) $(m+n)\ \vec{a} = m\vec{a} + n\vec{a}$ (iv) $m\ (\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}$

Self Practice Problems :

(1) Given a regular hexagon ABCDEF with centre O, show that
(i) $\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{OD} - \overrightarrow{OE}$ (ii) $\overrightarrow{EA} = 2\,\overrightarrow{OB} + \overrightarrow{OF}$ (iii) $\overrightarrow{AD} + \overrightarrow{EB} + \overrightarrow{FC} = 4\,\overrightarrow{AB}$

(2) Let ABCDEF be a regular hexagon. If $\overrightarrow{AD} = x\,\overrightarrow{BC}$ and $\overrightarrow{CF} = y\,\overrightarrow{AB}$ then find xy.

(3) The sum of the two unit vectors is a unit vector. Show that the magnitude of the their difference is $\sqrt{3}$.

Answers : (2) – 4

Addition of vectors :

(i) If two vectors $\vec{a}$ and $\vec{b}$ are represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$, then their sum $\vec{a}+\vec{b}$ is a vector represented by $\overrightarrow{OC}$, where OC is the diagonal of the parallelogram OACB.

(ii) $\vec{a}+\vec{b} = \vec{b}+\vec{a}$ **(commutative)** (iii) $(\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$ **(associative)**

(iv) $\vec{a}+\vec{0} = \vec{a} = \vec{0}+\vec{a}$ (v) $\vec{a}+(-\vec{a}) = \vec{0} = (-\vec{a})+\vec{a}$

(vi) $|\vec{a}+\vec{b}| \le |\vec{a}|+|\vec{b}|$ (vii) $|\vec{a}-\vec{b}| \ge ||\vec{a}|-|\vec{b}||$

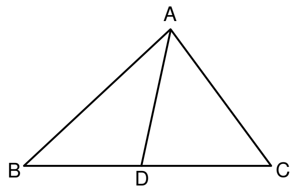
Example # 3 : The two sides of ΔABC are given by $\overrightarrow{AB} = 2\hat{i} + 4\hat{j} + 4\hat{k}$, $\overrightarrow{AC} = 2\hat{i} + 2\hat{j} + \hat{k}$. Then find the length of median through A.

Solution : Let D be mid point of BC

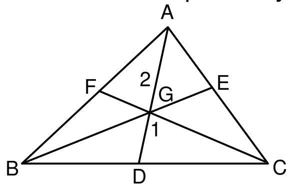
In ΔABC, $\overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$ $\Rightarrow$ $\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \overrightarrow{AD}$

$\Rightarrow$ $\frac{\overrightarrow{AB}+(\overrightarrow{AB}+\overrightarrow{BC})}{2} = \overrightarrow{AD}$

$\Rightarrow$ $\frac{\overrightarrow{AB}+\overrightarrow{AC}}{2} = \overrightarrow{AD}$ $\Rightarrow |\overrightarrow{AD}| = \left|\frac{4\hat{i}+6\hat{j}+5\hat{k}}{2}\right| = \frac{\sqrt{77}}{2}$



Example # 4 : In a triangle ABC, D, E, F are the mid-points of the sides BC, CA and AB respectively then prove that, $\overrightarrow{AD} = -(\overrightarrow{BE} + \overrightarrow{CF})$.



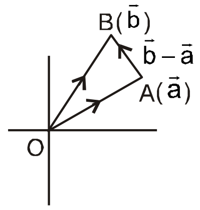
Solution : $\overrightarrow{AD} = 3\,\overrightarrow{GD} = 3.\,\frac{1}{2}\,(\overrightarrow{GB}+\overrightarrow{GC})$ where D is mid-point of BC

$$= \frac{3}{2}\left[\frac{2}{3}\overrightarrow{EB}+\frac{2}{3}\overrightarrow{FC}\right] = -(\overrightarrow{BE} + \overrightarrow{CF})$$

Position vector of a point:

Let O be a fixed origin, then the position vector of a point P is the vector $\overrightarrow{OP}$. If $\vec{a}$ and $\vec{b}$ are position vectors of two points A and B, then

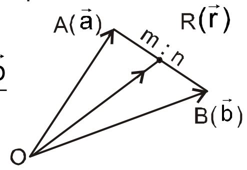
$\overrightarrow{AB} = \vec{b}-\vec{a}$ = position vector **(p.v.)** of B – position vector **(p.v.)** of A.



DISTANCE FORMULA

Distance between the two points $A(\vec{a})$ and $B(\vec{b})$ is AB = $\left|\vec{a}-\vec{b}\right|$

SECTION FORMULA



If $\vec{a}$ and $\vec{b}$ are the position vectors of two points A (x_1, y_1, z_1) and $B(x_2, y_2, z_2)$,

then the p.v. of a point R which divides AB in the ratio m: n is given by $\vec{r} = \dfrac{n\vec{a}+m\vec{b}}{m+n}$

Here R = $\left(\dfrac{nx_1+mx_2}{n+m},\dfrac{ny_1+my_2}{n+m},\dfrac{nz_1+mz_2}{n+m}\right)$

Note : Position vector of mid point M of AB is $\dfrac{\vec{a}+\vec{b}}{2}$. Here M = $\left(\dfrac{x_1+x_2}{2},\dfrac{y_1+y_2}{2},\dfrac{z_1+z_2}{2}\right)$

Example # 5 : Let O be the centre of a regular pentagon ABCDE and $\overrightarrow{OA} = \vec{a}$.
Then $\overrightarrow{AB}+2\overrightarrow{BC}+3\overrightarrow{CD}+4\overrightarrow{DE}+5\overrightarrow{EA}$ =

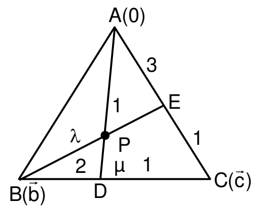
Solution : $\overrightarrow{OA}=\vec{a},\overrightarrow{OB}=\vec{b},\overrightarrow{OC}=\vec{c},\overrightarrow{OD}=\vec{d},\overrightarrow{OE}=\vec{e}$

$\overrightarrow{AB}+2\overrightarrow{BC}+3\overrightarrow{CD}+4\overrightarrow{DE}+5\overrightarrow{EA} = \left(\vec{b}-\vec{a}\right) + 2\left(\vec{c}-\vec{b}\right) + 3\left(\vec{d}-\vec{c}\right) + 4\left(\vec{e}-\vec{d}\right) + 5\left(\vec{a}-\vec{e}\right)$

$= 5\vec{a} - \left(\vec{a}+\vec{b}+\vec{c}+\vec{d}+\vec{e}\right) = 5\vec{a}$, (since $\vec{a}+\vec{b}+\vec{c}+\vec{d}++\vec{e} = 0$)

Example # 6 : In a triangle ABC, D and E are points on BC and AC respectively, such that BD = 2DC and AE = 3EC. Let P be the point of intersection of AD and BE. Find $\dfrac{BP}{PE}$ using vector method.

Solution : Let the position vectors of points B and C be respectively $\vec{b}$ and $\vec{c}$ referred to A as origin of reference.



Let $\dfrac{BP}{PE} = \lambda$ and $\dfrac{PD}{AP} = \mu$

$\overrightarrow{AD} = \dfrac{2\vec{c}+\vec{b}}{3}$, $\overrightarrow{AE} = \dfrac{3}{4}\vec{c} \Rightarrow \overrightarrow{AP} = \dfrac{\frac{3\lambda\vec{c}}{4}+\vec{b}}{\lambda+1} = \dfrac{\frac{2\vec{c}+\vec{b}}{3}}{\mu+1}$

comparing the coefficient of $\vec{b}$ & $\vec{c}$

$\dfrac{1}{\lambda+1} = \dfrac{1}{3(\mu+1)}$ and $\dfrac{3\lambda}{4(\lambda+1)} = \dfrac{2}{3(\mu+1)}$

solving above equations we get $\lambda = 8/3$

Self Practice Problems

(4) Express vectors $\overrightarrow{BC}$, $\overrightarrow{CA}$ and $\overrightarrow{AB}$ in terms of the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$

(5) If $\vec{a}, \vec{b}$ are position vectors of the points(1,–1),(–2, m), find the value of m for which $\vec{a}$ and $\vec{b}$ are collinear.

(6) The vertices P, Q and S of a ΔPQS have position vectors $\vec{p}, \vec{q}$ and $\vec{s}$ respectively.

(i) Find the position vector of $\vec{t}$ of point T in terms of $\vec{p}, \vec{q}$ and $\vec{s}$, such that ST : TM = 2 : 1 and M is mid-point of PQ.

(ii) If the parallelogram PQRS is now completed. Express , the position vector of the point R in terms of $\vec{p}, \vec{q}$ and $\vec{s}$

(7) In a quadrilateral ABCD, $\overrightarrow{AB} = \vec{p}$, $\overrightarrow{BC} = \vec{q}$, $\overrightarrow{DA} = \vec{p} - \vec{q}$. If E is the mid point of BC and F is the point on DE such that DF = $\frac{4}{5}$ DE. Show that the points. A,F,C are collinear.

(8) Point L, M, N divide the sides BC, CA, AB of ΔABC in the ratios 1 : 4, 3 : 2, 3 : 7 respectively. Prove that $\overrightarrow{AL} + \overrightarrow{BM} + \overrightarrow{CN}$ is a vector parallel to $\overrightarrow{CK}$, when K divides AB in the ratio 1 : 3.

Answers : (4) $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$, $\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ (5) m = 2

(6) (i) $\vec{t} = \frac{1}{3}(\vec{p} + \vec{q} + \vec{s})$ (ii) $\vec{r} = (\vec{q} - \vec{p} + \vec{s})$

Distance formula

Distance between any two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given as $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

Distance of a point P from coordinate axes

Let PA, PB and PC are distances of the point P(x, y, z) from the coordinate axes OX, OY and OZ respectively then PA = $\sqrt{y^2 + z^2}$, PB = $\sqrt{z^2 + x^2}$, PC = $\sqrt{x^2 + y^2}$

Example # 7 : Find the locus of a point which is equidistance from A (0,2,3) and B (2, –2, 1).

Solution : let P (x, y, z) be any point which is equidistance from A (0,2,3) and B (2, – 2, 1)

PA = PB

$\Rightarrow \sqrt{(x-0)^2 + (y-2)^2 + (z-3)^2} = \sqrt{(x-2)^2 + (y+2)^2 + (z-1)^2} \Rightarrow x - 2y - z + 1 = 0$

Example # 8 : Find the locus of a point which moves such that the sum of its distances from points A(0, 0, – α) and B(0, 0, α) is constant.

Solution : Let the variable point whose locus is required be P(x, y, z)

Given PA + PB = constant = 2a (say)

$\therefore \quad \sqrt{(x-0)^2 + (y-0)^2 + (z+\alpha)^2} + \sqrt{(x-0)^2 + (y-0)^2 + (z-\alpha)^2} = 2a$

$\Rightarrow \quad \sqrt{x^2 + y^2 + (z+\alpha)^2} = 2a - \sqrt{x^2 + y^2 + (z-\alpha)^2}$

$\Rightarrow \quad x^2 + y^2 + z^2 + \alpha^2 + 2z\alpha = 4a^2 + x^2 + y^2 + z^2 + \alpha^2 - 2z\alpha - 4a\sqrt{x^2 + y^2 + (z-\alpha)^2}$

$\Rightarrow \quad 4z\alpha - 4a^2 = -4a\sqrt{x^2 + y^2 + (z-\alpha)^2} \quad \Rightarrow \quad \frac{z^2\alpha^2}{a^2} + a^2 - 2z\alpha = x^2 + y^2 + z^2 + \alpha^2 - 2z\alpha$

or, $\quad x^2 + y^2 + z^2\left(1 - \frac{\alpha^2}{a^2}\right) = a^2 - \alpha^2 \quad \Rightarrow \quad \frac{x^2}{a^2 - \alpha^2} + \frac{y^2}{a^2 - \alpha^2} + \frac{z^2}{a^2} = 1$

This is the required locus.

Self practice problems :

(9) One of the vertices of a cuboid is (0, 2, –1) and the edges from this vertex are along the positive x-axis, positive y-axis and positive z-axis respectively and are of lengths 2, 2, 3 respectively find out the vertices.

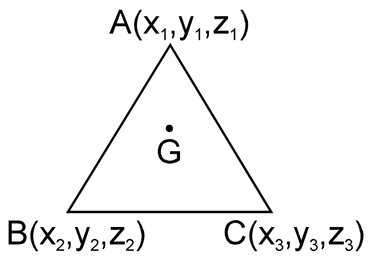
(10) Show that the points (0, 4, 1), (2, 3, –1), (4, 5, 0) and (2, 6, 2) are the vertices of a square.

(11) Find the locus of point P if $AP^2 – BP^2 = 20$, where A ≡ (2, –1, 3) and B ≡ (–1, – 2, 1).

Answers : (9) (2,2, –1), (2, 4, –1), (2, 4, 2), (2, 2, 2), (0, 2, 2), (0, 4, 2), (0, 4, –1).
(11) x + y + 2z = 6

Centroid of a triangle

$$G \equiv \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$



Incentre of triangle ABC

$$I \equiv \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}, \frac{az_1 + bz_2 + cz_3}{a+b+c}\right)$$ Where AB = c, BC = a, CA = b

Example # 9 : Show that the points A(2, 3, 4), B(–1, 2, –3) and C(–4, 1, –10) are collinear. Also find the ratio in which C divides AB.

Solution : Given A ≡ (2, 3, 4), B ≡ (–1, 2, –3), C ≡ (– 4, 1, –10).

Let C divide AB internally in the ratio k : 1, then

$$C \equiv \left(\frac{-k+2}{k+1}, \frac{2k+3}{k+1}, \frac{-3k+4}{k+1}\right) \quad \therefore \quad \frac{-k+2}{k+1} = -4 \quad \Rightarrow \quad 3k = -6 \Rightarrow \quad k = -2$$

For this value of k, $\frac{2k+3}{k+1} = 1$, and $\frac{-3k+4}{k+1} = -10$

Since k < 0, therefore C divides AB externally in the ratio 2 : 1 and points A, B, C are collinear.

Example # 10 : The vertices of a triangle are A(5, 4, 6), B(1, –1, 3) and C(4, 3, 2). The internal bisector of ∠BAC meets BC in D. Find AD.

Solution : AB = $\sqrt{4^2 + 5^2 + 3^2} = 5\sqrt{2}$

AC = $\sqrt{1^2 + 1^2 + 4^2} = 3\sqrt{2}$

Since AD is the internal bisector of BAC

∴ $\frac{BD}{DC} = \frac{AB}{AC} = \frac{5}{3}$ ∴ D divides BC internally in the ratio 5 : 3

∴ $D \equiv \left(\frac{5\times4 + 3\times1}{5+3}, \frac{5\times3 + 3(-1)}{5+3}, \frac{5\times2 + 3\times3}{5+3}\right)$ or, $D = \left(\frac{23}{8}, \frac{12}{8}, \frac{19}{8}\right)$

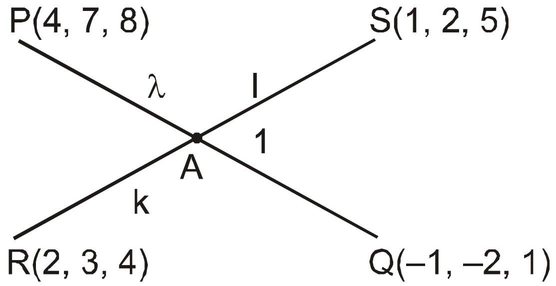
∴ AD = $\sqrt{\left(5 - \frac{23}{8}\right)^2 + \left(4 - \frac{12}{8}\right)^2 + \left(6 - \frac{19}{8}\right)^2} = \frac{\sqrt{1530}}{8}$ unit

Example # 11 : If the points P, Q, R, S are (4, 7, 8), (– 1, – 2, 1), (2, 3, 4) and (1,2,5) respectively, show that PQ and RS intersect. Also find the point of intersection.

Solution : Let the lines PQ and RS intersect at point A.

Let A divide PQ in the ratio λ : 1, ($\lambda \neq -1$) then $A \equiv \left(\frac{-\lambda+4}{\lambda+1}, \frac{-2\lambda+7}{\lambda+1}, \frac{\lambda+8}{\lambda+1}\right)$. .... (1)

Let A divide RS in the ratio k : 1, then $A \equiv \left(\frac{k+2}{k+1}, \frac{2k+3}{k+1}, \frac{5k+4}{k+1}\right)$ ..... (2)



From (1) and (2), we have,

$\frac{-\lambda+4}{\lambda+1} = \frac{k+2}{k+1} \Rightarrow -\lambda k - \lambda + 4k + 4 = \lambda k + 2\lambda + k + 2 \Rightarrow 2\lambda k + 3\lambda - 3k - 2 = 0$ .....(3)

$\frac{-2\lambda+7}{\lambda+1} = \frac{2k+3}{k+1} \Rightarrow -2\lambda k - 2\lambda + 7k + 7 = 2\lambda k + 3\lambda + 2k + 3 \Rightarrow 4\lambda k + 5\lambda - 5k - 4 = 0$ .....(4)

$\frac{\lambda+8}{\lambda+1} = \frac{5k+4}{k+1}$ ..... (5)

Multiplying equation (3) by 2, and subtracting from equation (4), we get $-\lambda + k = 0$ or, $\lambda = k$

Putting $\lambda = k$ in equation (3), we get $2\lambda^2 + 3\lambda - 3\lambda - 2 = 0 \Rightarrow \lambda = 1 = k$

Clearly $\lambda = k = 1$ satisfies eqn. (5), hence our assumption is correct.

$\therefore$ $A \equiv \left(\frac{-1+4}{2}, \frac{-2+7}{2}, \frac{1+8}{2}\right)$ or, $A \equiv \left(\frac{3}{2}, \frac{5}{2}, \frac{9}{2}\right)$.

Self practice problems :

(12) Find the ratio in which yz plane divides the line joining the points A (4, 3, 5) and B (7, 4, 5).

(13) Find the co-ordinates of the foot of perpendicular drawn from the point A(1, 2, 1) to the line joining the point B(1, 4, 6) and C(5, 4, 4).

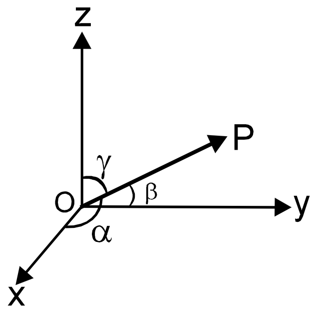
(14) Two vertices of a triangle are (4, –6, 3) and (2, –2, 1) and its centroid is $\left(\frac{8}{3}, -1, 2\right)$. Find the third vertex.

(15) Show that $\left(\frac{3}{2}, \frac{7}{2}, \frac{1}{2}\right)$ is the circumcentre of the triangle whose vertices are A (2, 3, 2), B (0, 4, 1) and C (3, 3, 0) and hence find its orthocentre.

Answers : (12) 4 : 7 Externally (13) (3, 4, 5) (14) (2, 5, 2) (15) (2, 3, 2)

Direction cosines and direction ratios

(i) **Direction cosines :** Let α, β, γ be the angles which a directed line makes with the positive directions of the axes of x, y and z respectively, then cos α, cosβ, cos γ are called the direction cosines of the line. The direction cosines are usually denoted by ℓ, m, n.



Thus $\ell = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

(ii) If ℓ, m, n be the direction cosines of a line, then $\ell^2 + m^2 + n^2 = 1$

(iii) **Direction ratios :** Let a, b, c be proportional to the direction cosines ℓ, m, n then a, b, c are called the direction ratios.

If a, b, c, are the direction ratios of any line L, then $a\hat{i} + b\hat{j} + c\hat{k}$ will be a vector parallel to the line L.

If ℓ, m, n are direction cosines of line L, then $\ell\ \hat{i} + m\ \hat{j} + n\ \hat{k}$ is a unit vector parallel to the line L.

(iv) If ℓ, m, n be the direction cosines and a, b, c be the direction ratios of a vector, then (ℓ, m, n)

$$= \left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right) \text{or} \left(\frac{-a}{\sqrt{a^2+b^2+c^2}}, \frac{-b}{\sqrt{a^2+b^2+c^2}}, \frac{-c}{\sqrt{a^2+b^2+c^2}}\right)$$

(v) If the coordinates P and Q are (x_1, y_1, z_1) and (x_2, y_2, z_2), then the direction ratios of line PQ are, $a = x_2 - x_1$, $b = y_2 - y_1$ & $c = z_2 - z_1$ and the direction cosines of line PQ are $\ell = \frac{x_2 - x_1}{|PQ|}$, $m = \frac{y_2 - y_1}{|PQ|}$ and $n = \frac{z_2 - z_1}{|PQ|}$.

Example # 12 : If a line makes angle α, β, γ with the co-ordinate axes. Then find the value of $\sum \frac{\cos 3\alpha}{\cos\alpha}$.

Solution : $\sum \frac{\cos 3\alpha}{\cos\alpha} = \sum \frac{4\cos^3\alpha - 3\cos\alpha}{\cos\alpha} = \Sigma(4\cos^2\alpha - 3)$

$= 4(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) - 3 - 3 - 3 = 4 - 9 = -5$ **Ans.** -5

Example # 13 : If the direction ratios of two lines are given by $mn - 4n\ell + 3\ell m = 0$ and $\ell + 2m + 3n = 0$ then find the direction ratios of the lines.

Solution : Eliminating ℓ we have $m = \pm\sqrt{2}\ n$ $\therefore \frac{\ell}{-2\sqrt{2}-3} = \frac{m}{\sqrt{2}} = \frac{n}{1}$ & $\frac{\ell}{2\sqrt{2}-3} = \frac{m}{-\sqrt{2}} = \frac{n}{1}$

Ans. $((-2\sqrt{2}-3)\lambda, \sqrt{2}\lambda, \lambda), (2\sqrt{2}-3)\lambda\ ,\ \sqrt{2}\lambda - \lambda$ where $\lambda \in R - \{0\}$

Self practice problems:

(16) Find the direction cosines of a line lying in the xy plane and making angle 30° with x-axis.

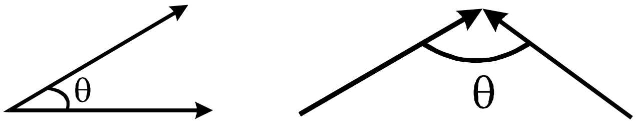
(17) A line makes an angle of 60° with each of x and y axes, find the angle which this line makes with z-axis.

(18) A plane intersects the co-ordinates axes at point A(2, 0, 0), B(0, 4, 0), C(0, 0, 6) ; O is origin. Find the direction ratio of the line joining the vertex B to the centroid of face ABC.

Answers : (16) $\ell = \frac{\sqrt{3}}{2}$, $m = \pm\frac{1}{2}$, $n = 0$ (17) 45° (18) $\frac{2}{3}, -\frac{8}{3}, 2$

Angle between two vectors :

It is the smaller angle formed when the initial points or the terminal points of the two vectors are brought together. Note that $0^\circ \le \theta \le 180^\circ$.



Scalar product (Dot Product) of two vectors :

$$\vec{a}\ .\ \vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta,\ (0 \le \theta \le \pi)$$

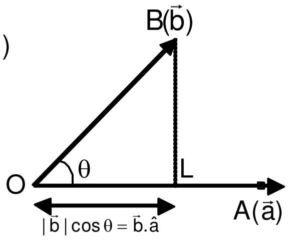
Note: (a) If θ is acute, then $\vec{a}\ .\ \vec{b} > 0$ and if θ is obtuse, then $\vec{a}\ .\ \vec{b} < 0$.

(b) $\vec{a}\ .\ \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$ $(\vec{a} \ne 0,\ \vec{b} \ne 0)$

(c) Maximum value of $\vec{a}.\vec{b}$ is $|\vec{a}|\,|\vec{b}|$ (d) Minimum value of $\vec{a}.\vec{b}$ is $-|\vec{a}|\,|\vec{b}|$

Geometrical interpretation of scalar product :

As shown in Figure, projection of vector $\overrightarrow{OB}$ (or $\vec{b}$) along vector $\overrightarrow{OA}$ (or $\vec{a}$) is $OL = |\vec{b}|\ \cos\theta = \frac{\vec{b}.\vec{a}}{|\vec{a}|} = \vec{b}.\hat{a}$



Properties of Dot Product

(i) Projection of $\vec{a}$ on $\vec{b}$ $= \frac{\vec{a}\ .\ \vec{b}}{|\vec{b}|}$

(ii) $\vec{a}\ .\ \vec{b} = \vec{b}\ .\ \vec{a}$ **(commutative)**

(iii) $\vec{a}\ .\ (\vec{b}+\vec{c}) = \vec{a}\ .\ \vec{b} + \vec{a}\ .\ \vec{c}$ **(distributive)**

(iv) $(m\vec{a})\ .\ \vec{b} = \vec{a}\ .(m\vec{b}) = m(\vec{a}\ .\ \vec{b})$, where m is a scalar.

(v) $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1;\ \hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$

(vi) $\vec{a}\ .\ \vec{a} = |\vec{a}|^2 = \vec{a}^2$

(vii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a}\ .\ \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, $|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$

(viii) $\left|\vec{a} \pm \vec{b}\right| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 \pm 2|\vec{a}||\vec{b}|\cos\theta}$, where θ is the angle between the vectors

Example # 14 : Find the value of p for which the vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ are

(i) perpendicular (ii) parallel

Solution : (i) $\vec{a} \perp \vec{b} \Rightarrow \vec{a}\ .\ \vec{b} = 0 \Rightarrow \left(3\hat{i} + 2\hat{j} + 9\hat{k}\right) . \left(\hat{i} + p\hat{j} + 3\hat{k}\right) = 0$

$\Rightarrow \quad 3 + 2p + 27 = 0 \quad \Rightarrow \quad p = -15$

(ii) vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ are parallel iff

$$\frac{3}{1} = \frac{2}{p} = \frac{9}{3} \quad \Rightarrow \quad 3 = \frac{2}{p} \quad \Rightarrow \quad p = \frac{2}{3}$$

Example # 15 : If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three vectors such that each is inclined at an angle $\pi/3$ with the other two and $|\vec{a}| = 1$, $|\vec{b}| = 2$, $|\vec{c}| = 3$, then find the scalar product of the vectors $2\vec{a} + 3\vec{b} - 5\vec{c}$ and $4\vec{a} - 6\vec{b} + 10\vec{c}$.

Solution : Dot products is $8^2\vec{a} - 18^2\vec{b} - 50^2\ \vec{c} + \vec{a}.\vec{b}\ (-12 + 12) + \vec{b}.\vec{c}\ (30 + 30) + \vec{c}\ .\ \vec{a}\ (20 - 20)$

$= 8 - 18\ (4) - 50(9) + 60\ \left(2.3\cos\frac{\pi}{3}\right) = 188 - 522 = -334$

Example # 16 : Find the values of x for which the angle between the vectors
$\vec{a} = 2x^2\hat{i} + 4x\hat{j} + \hat{k}$ and $\vec{b} = 7\hat{i} - 2\hat{j} + x\hat{k}$ is obtuse.

Solution : The angle θ between vectors $\vec{a}$ and $\vec{b}$ is given by $\cos\theta = \dfrac{\vec{a}\ .\ \vec{b}}{|\vec{a}||\vec{b}|}$

Now, θ is obtuse $\Rightarrow \cos\theta < 0 \Rightarrow \dfrac{\vec{a}\ .\ \vec{b}}{|\vec{a}||\vec{b}|} < 0 \Rightarrow \vec{a}\ .\ \vec{b} < 0 \qquad [\because |\vec{a}|, |\vec{b}| > 0]$

$\Rightarrow 14x^2 - 8x + x < 0 \Rightarrow 7x\ (2x - 1) < 0 \Rightarrow x(2x - 1) < 0 \Rightarrow 0 < x < \dfrac{1}{2}$

Hence, the angle between the given vectors is obtuse if $x \in (0, 1/2)$

Example # 17 : If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $= 2\vec{a} - \hat{j} + 3\hat{k}$, then find

(i) Component of $\vec{b}$ along $\vec{a}$. (ii) Component of $\vec{b}$ in plane of $\vec{a}$ & $\vec{b}$ but $\perp$ to $\vec{a}$.

Solution : (i) Component of $\vec{b}$ along $\vec{a}$ is $\left(\dfrac{\vec{a}\ .\ \vec{b}}{|\vec{a}|^2}\right)\vec{a}$; Here $\vec{a}.\vec{b} = 2 - 1 + 3 = 4$ and $|\vec{a}|^2 = 3$

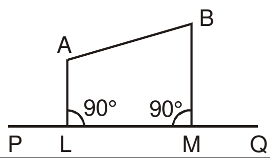
Hence $\left(\dfrac{\vec{a}\ .\ \vec{b}}{|\vec{a}|^2}\right)\vec{a} = \dfrac{4}{3}\vec{a} = \dfrac{4}{3}(\hat{i} + \hat{j} + \hat{k})$

(ii) Component of $\vec{b}$ in plane of $\vec{a}$ & $\vec{b}$ but $\perp$ to $\vec{a}$ is $\vec{b} - \left(\dfrac{\vec{a}\ .\ \vec{b}}{|\vec{a}|^2}\right)\vec{a}\ . = \dfrac{1}{3}\left(2\hat{i} - 7\hat{j} + 5\hat{k}\right)$

Example # 18 : Find the projection of the line joining A(1, 2, 3) and B(–1, 4, 2) on the line having direction ratios 2, 3, –6.

Solution : $\overrightarrow{AB} = -2\hat{i} + 2\hat{j} - \hat{k}$

Projection of $\overrightarrow{AB} = -2\hat{i} + 2\hat{j} - \hat{k}$ on $2\hat{i} + 3\hat{j} - 6\hat{k}$ is $\dfrac{-4 + 6 + 6}{\sqrt{4 + 9 + 36}} = \dfrac{8}{7}$



Self Practice Problems :

(19) If $\vec{a}$ and $\vec{b}$ are unit vectors and θ is angle between them, prove that $\tan\frac{\theta}{2} = \frac{|\vec{a}-\vec{b}|}{|\vec{a}+\vec{b}|}$.

(20) Find the values of x for which the angle between the vectors $\vec{a} = 2x\,\hat{i}+4\hat{j}+3\hat{k}$ and $\vec{b} = 2\hat{i}-3\hat{j}+x\hat{k}$ is 90º

(21) if a, b, c are the pth, qth, rth terms of a HP then find the angle between the vectors $= (q-r)\,\hat{i} + (r-q)\,\hat{j} + (p-q)\,\hat{k}$ and $\vec{v} = \frac{1}{a}\,\hat{i} + \frac{1}{b}\,\hat{j} + \frac{1}{c}\,\hat{k}$.

(22) The points O, A, B, C, D are such that $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}, \overrightarrow{OC} = 2\vec{a}+3\vec{b}, \overrightarrow{OD} = \vec{a}+2\vec{b}$
Given that the length of $\overrightarrow{OA}$ is three times the length of $\overrightarrow{OB}$. Show that $\overline{BD}$ and $\overline{AC}$ are perpendicular.

(23) ABCD is a tetrahedron and G is the centroid of the base BCD. Prove that
$AB^2 + AC^2 + AD^2 = GB^2 + GC^2 + GD^2 + 3GA^2$

(24) A (2, 3, –2), B (1, 5, 4,), C(0, –1, 2) D (4, 0, 3). Find the projection of line segment AB on CD line.

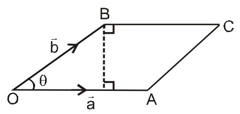
(25) The projections of a directed line segment on co-ordinate axes are 3, 4, –12. Find its length and direction cosines.

Answers : (20) x = 12/7 (21) $\pi/2$ (24) $\frac{2\sqrt{2}}{3}$ (25) 13, $\frac{3}{13}, \frac{4}{13}, \frac{-12}{13}$

Vector product (Cross Product) of two vectors:

(i) If $\vec{a}$, $\vec{b}$ are two vectors and θ is the angle between them, then $\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\,\hat{n}$, where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$ and $\hat{n}$ forms a right handed screw system.

(ii) Geometrically $|\vec{a} \times \vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ and $\vec{b}$.



(iii) $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ **(not commutative)**

(iv) $(m\vec{a}) \times \vec{b} = \vec{a} \times (m\vec{b}) = m(\vec{a} \times \vec{b})$, where m is a scalar.

(v) $\vec{a} \times (\vec{b}+\vec{c}) = (\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c})$ **(distributive)**

(vi) $\vec{a}\times\vec{b} = \vec{0} \Leftrightarrow \vec{a}$ and $\vec{b}$ are parallel (collinear) $(\vec{a} \neq 0\ ,\ \vec{b} \neq 0)$ i.e. $\vec{a} = K\vec{b}$, where K is a scalar.

(vii) $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$; $\hat{i}\times\hat{j} = \hat{k},\ \hat{j}\times\hat{k} = \hat{i},\ \hat{k}\times\hat{i} = \hat{j}$

(viii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

(x) A vector of magnitude 'r' and perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\pm\dfrac{r(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$

(xii) If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are the position vectors of 3 points A, B and C respectively, then the vector area of $\Delta ABC = \dfrac{1}{2}(\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a})$. The points A, B and C are collinear if

$\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} = \vec{0}$

(xiii) Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ and $\vec{d}_2$ is given by $\dfrac{1}{2}\left|\vec{d}_1 \times \vec{d}_2\right|$

Example # 19 : Given $\vec{a} = \hat{i} + \hat{j} - \hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = -\hat{i} + 2\hat{j} - \hat{k}$, then find a unit vector perpendicular to both $\vec{a} + \vec{b}$ and $\vec{b} + \vec{c}$.

Solution : $\vec{\alpha} \times \vec{\beta}$ is $\perp$ to both $\vec{\alpha}$ and $\vec{\beta}$

$\vec{a} + \vec{b} = 3\hat{j}$, $\vec{b} + \vec{c} = -2\hat{i} + 4\hat{j}$

$\therefore$ $3\hat{j} \times (-2\hat{i} + 4\hat{j})$ is $\perp$ to both or $-6\hat{j} \times \hat{i} = 6\hat{k}$ $\therefore$ hence a unit vector is $\hat{k}$.

Example # 20 : If $\vec{\alpha} = 2\hat{i} + 3\hat{j} - \hat{k}$, $\vec{\beta} = -\hat{i} + 2\hat{j} - 4\hat{k}$, $\vec{\gamma} = \hat{i} + \hat{j} + \hat{k}$, then find value $(\vec{\alpha} \times \vec{\beta}).(\vec{\alpha} \times \vec{\gamma})$.

Solution : $\vec{\alpha} \times \vec{\beta} = -10\hat{i} + 9\hat{j} + 7\hat{k}$ and $\vec{\alpha} \times \vec{\gamma} = 4\hat{i} - 3\hat{j} - \hat{k}$ their dot product $= -40 - 27 - 7 = -74$

Example # 21 : Let $\overrightarrow{OA} = \vec{a} + 3\vec{b}$, $\overrightarrow{OB} = 5\vec{a} + 4\vec{b}$ and $\overrightarrow{OC} = 2\vec{a} - \vec{b}$ where O is origin. Let p denote the area of the quadrilateral OABC and q denote the area of the parallelogram with OA and OC as adjacent sides. Find $\dfrac{p}{q}$.

Solution : We have, p = Area of the quadrilateral OABC

$\Rightarrow$ $p = \dfrac{1}{2}|\overrightarrow{OB}\times\overrightarrow{AC}| = \dfrac{1}{2}|\overrightarrow{OB}\times(\overrightarrow{OC}-\overrightarrow{OA})|$ $\Rightarrow$ $p = \dfrac{1}{2}|(5\vec{a}+4\vec{b}) \times (4\vec{b}-\vec{a})|$

$\Rightarrow$ $p = \dfrac{1}{2}|20(\vec{a}\times\vec{b}) - 4(\vec{b}\times\vec{a})| = 12\ |\vec{a}\times\vec{b}|$ .....(i)

and q = Area of the parallelogram with OA and OC as adjacent sides

$\Rightarrow$ $q = |\overrightarrow{OA}\times\overrightarrow{OC}| = |(\vec{a}\times3\vec{b})\times(2\vec{a}-\vec{b})| = 7|\vec{a}\times\vec{b}|$ .....(ii)

From (i) and (ii), we get $\dfrac{p}{q} = \dfrac{12}{7}$

Self Practice Problems :

(26) If $\vec{p}$ and $\vec{q}$ are unit vectors forming an angle of 30º. Find the area of the parallelogram having $\vec{a} = \vec{p} + 2\vec{q}$ and $\vec{b} = 2\vec{p} + \vec{q}$ as its diagonals.

(27) ABC is a triangle and EF is any straight line parallel to BC meeting AC, AB in E, F respectively. If BR and CQ be drawn parallel to AC, AB respectively to meet EF in R and Q respectively, prove that Δ ARB = ΔACQ.

Answers : (26) 3/4 sq. units

A LINE

Equation of a line

(i) Vector equation: Vector equation of a straight line passing through a fixed point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$ where λ is a scalar.

(ii) Vector equation of a straight line passing through two points with position vectors $\vec{a}$ & $\vec{b}$ is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$.

(iii) The equation of a line passing through the point (x_1, y_1, z_1) and having direction ratios a, b, c is $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = r$. This form is called symmetric form. A general point on the line is given by $(x_1 + ar, y_1 + br, z_1 + cr)$.

(iv) The equation of the line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

(v) Reduction of cartesian form of equation of a line to vector form & vice versa

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \Leftrightarrow = \vec{r}\ (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda(a\hat{i} + b\hat{j} + c\hat{k}).$$

(vi) The equations of the bisectors of the angles between the lines $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{a} + \mu\vec{c}$ are : $\vec{r} = \vec{a} + t\left(\hat{b} + \hat{c}\right)$ and $\vec{r} = \vec{a} + p\left(\hat{c} - \hat{b}\right)$.

Example # 22 : Find the equation of the line through the points (4, –5, 8) and (–1, 2, 7) in vector form as well as in cartesian form.

Solution : Let A ≡ (4, –5, 8) , B ≡ (–1, 2, 7)

Now $\vec{a} = \vec{OA} = 4\hat{i} - 5\hat{j} + 8\hat{k}$ and $\vec{b} = \overline{OB} = -\hat{i} + 2\hat{j} + 7\hat{k}$

Equation of the line through A($\vec{a}$) and B($\vec{b}$) is $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$

or $\vec{r} = 4\hat{i} - 5\hat{j} + 8\hat{k} + t(-5\hat{i} + 7\hat{j} - \hat{k})$ ..... (1)

Equation of AB in cartesian form is $\frac{x-4}{5} = \frac{y+5}{-7} = \frac{z-8}{1}$

Example # 23 : Find the equation of the line passing through point (1, 0, 2) having direction ratio 3, –1, 5. Prove that this line passes through (4, – 1, 7).

Solution. equation of line is $\frac{x-1}{3} = \frac{y-0}{-1} = \frac{z-2}{5}$,

Now $\frac{4-1}{3} = \frac{-1-0}{-1} = \frac{7-2}{5} = 1$, so line passes through point (4, –1,7)

Example # 24 : Find the equation of the line drawn through point (–1, 7, 0) to meet at right angles the line $\frac{x-2}{2} = \frac{y+3}{-1} = \frac{z-1}{2}$

Solution : Given line is $\frac{x-2}{2} = \frac{y+3}{-1} = \frac{z-1}{2}$ ..... (1)

Let P ≡ (–1, 7, 0)

Co-ordinates of any point on line (1) may be taken as Q ≡ (2r + 2, – r – 3, 2r + 1)

Direction ratios of PQ are 2r + 3, – r – 10, 2r + 1

Direction ratios of line AB are 2, – 1, 2

Since PQ $\perp$ AB

$\therefore$ 2(2r + 3) + (– r – 10) (–1) + 2 (2r + 1) = 0 $\Rightarrow$ r = – 2

Therefore, direction ratios of PQ are – 1 , – 8, – 3

Equation of line PQ is $\frac{x+1}{1} = \frac{y-7}{8} = \frac{z}{3}$

Example # 25 : A line passes through the point $3\hat{i}$ and is parallel to the vector $-\hat{i}+\hat{j}+\hat{k}$ and another line passes through the point $\hat{i}+\hat{j}$ and is parallel to the vector $\hat{i}+\hat{k}$, then find the point of intersection of lines.

Solution : A point on the first line is $3\hat{i}+s(-\hat{i}+\hat{j}+\hat{k})$ ....(i)

A point on the second line is $\hat{i}+\hat{j}+t(\hat{i}+\hat{k})$ ....(ii)

At the point of intersection (i) and (ii) are same.

$\therefore$ 3 – s = 1 + t, s = 1, s = t $\therefore$ s = t = 1

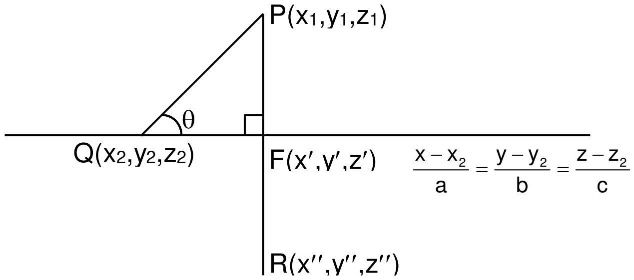
hence the point is $3\hat{i}+(-\hat{i}+\hat{j}+\hat{k})$ = $2\hat{i}+\hat{j}+\hat{k}$ Ans. (2,1,1)

Self practice problems:

(28) Find the equation of the line parallel to line $\frac{x-3}{4} = \frac{y+1}{-1} = \frac{z-7}{5}$ and passing through the point (2, 3, –2).

Answers : (28) $\frac{x-2}{4} = \frac{y-3}{1} = \frac{z+2}{5}$

Foot, Reflection, length of perpendicular from a point to a line :



Let L ≡ $\frac{x-x_2}{a} = \frac{y-y_2}{b} = \frac{z-z_2}{c}$ is a given line and P(x_1, y_1, z_1) is given point as shown in figure.

Let F(x′, y′, z′) = (ar+x_2, br+ y_2, cr+ z_2) ….(1) be the foot of the point P (x_1, y_1, z_1) with respect to the line L. Apply $\overrightarrow{PF}.(a\hat{i}+b\hat{j}+c\hat{k})=0$ we get ‘r’. Now put this value of ‘r’ in (1) we get F

Now for calculating the reflection R(x″, y″, z″) of the point P (x_1, y_1, z_1) with respect to the line L, apply midpoint formula (midpoint of P & R is F)

$$PF = PQ \sin\theta = \left|\frac{\overline{PF}.(a\hat{i}+b\hat{j}+c\hat{k})}{\sqrt{a^2+b^2+c^2}}\right|$$

Example # 26 : Find the length of the perpendicular from P (2, – 3, 1) to the line $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z+2}{-1}$.

Solution : Co-ordinates of any point on given line may be taken as Q ≡ (2r – 1, 3r + 3, – r – 2)
Direction ratios of PQ are 2r – 3, 3r + 6, – r – 3
Direction ratios of AB are 2, 3, – 1
Since PQ ⊥ AB

$\therefore$ 2 (2r – 3) + 3 (3r + 6) – 1 (– r – 3) = 0 ⇒ 14r + 15 = 0 ⇒ $r = \frac{-15}{14}$

$\therefore$ $Q \equiv \left(\frac{-22}{7}, \frac{-3}{14}, \frac{-13}{14}\right)$ $\therefore$ $PQ = \sqrt{\frac{531}{14}}$ units.

Self practice problems :

(29) Find the length and foot of perpendicular drawn from point (2,3,4) to the line $\frac{x-4}{-2} = \frac{y}{6} = \frac{z-1}{-3}$. Also find the image of the point in the line.

Answers : (29) $3\sqrt{5}$, N ≡ (2, 6, –2), I ≡ (2, 9,–8)

Angle between two line :

If two lines have direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 respectively, then we can consider two vectors parallel to the lines as $a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$ and angle between them can be given as.

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}.$$

(i) The lines will be perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

(ii) The lines will be parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Example # 27 : What is the angle between the lines whose direction cosines are

$$-\frac{\sqrt{3}}{4}, \frac{1}{4}, -\frac{\sqrt{3}}{2} \quad \text{and} \quad -\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}$$

Solution : Let θ be the required angle, then $\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$

$$= \left(-\frac{\sqrt{3}}{4}\right)\left(-\frac{\sqrt{3}}{4}\right)+\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)+\left(-\frac{\sqrt{3}}{2}\right).\left(\frac{\sqrt{3}}{2}\right) = \frac{3}{16}+\frac{1}{16}-\frac{3}{4} = -\frac{1}{2} \Rightarrow \theta = 120°,$$

Example # 28 : P is a point on line $\vec{r} = 5\hat{i}+7\hat{j}-2\hat{k}+s(3\hat{i}-\hat{j}+\hat{k})$ and Q is a point on the line $\vec{r} = -3\hat{i}+3\hat{j}+6\hat{k} + t\left(-3\hat{i}+2\hat{j}+4\hat{k}\right)$. If $\overrightarrow{PQ}$ is parallel to the vector, $2\hat{i}+7\hat{j}-5\hat{k}$, find P and Q

Solution : $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$ is parallel to $2\hat{i}+7\hat{j}-5\hat{k}$

$\therefore$ $-8\hat{i}-4\hat{j}+8\hat{k}+t(-3\hat{i}+2\hat{j}+4\hat{k})-s\left(3\hat{i}-\hat{j}+\hat{k}\right)$ = 2α = –8 – 3t – 3s

7α = –4 + 2t + s ⇒ –5α = 8 + 4t – s

solving, α = t = s = –1

$\therefore$ P = (5, 7, –2) – (3, –1, 1) = (2, 8, –3) ⇒ Q = (–3, 3, 6) – (–3, 2, 4) = (0, 1, 2)

Self practice problems :

(30) Find the angle between the lines whose direction cosines are given by $\ell + m + n = 0$ and $\ell^2 + m^2 - n^2 = 0$

(31) Let P (6, 3, 2), Q (5, 1, 4), R (3, 3, 5) are vertices of a Δ find $\angle$Q.

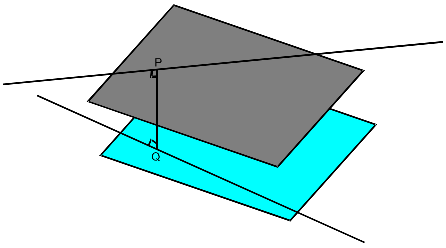
(32) Show that the direction cosines of a line which is perpendicular to the lines having directions cosines ℓ_1 m_1 n_1 and ℓ_2 m_2 n_2 respectively are proportional to

$m_1n_2 - m_2n_1$, $n_1\ell_2 - n_2\ell_1$, $\ell_1m_2 - \ell_2m_1$

Answers : (30) 60° (31) 90°

Skew Lines :

Lines in space which do not intersect and are also not parallel are called **skew line**.



If lines $\vec{r} = \vec{a} + \lambda_1\vec{p}$ & $\vec{r} = \vec{b} + \lambda_2\vec{q}$ are skew lines then $(\vec{b} - \vec{a}) \cdot (\vec{p} \times \vec{q}) \neq 0$

If lines are not skew lines then they are coplanar which means if $(\vec{b} - \vec{a}) \cdot (\vec{p} \times \vec{q}) = 0$, then lines are coplanar.

Shortest distance between two lines

(i) **Shortest distance (d) between lines** $\vec{r} = \vec{a} + \lambda_1\vec{p}$ & $\vec{r} = \vec{b} + \lambda_2\vec{q}$

is $d = \left| \dfrac{(\vec{b} - \vec{a}) \cdot (\vec{p} \times \vec{q})}{|\vec{p} \times \vec{q}|} \right|$

(ii) Shortest distance (d) between two skew lines $\dfrac{x-\alpha}{\ell} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n}$ and $\dfrac{x-\alpha'}{\ell'} = \dfrac{y-\beta'}{m'} = \dfrac{z-\gamma'}{n'}$

is $d = \begin{vmatrix} \alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix} \div \sqrt{\sum (mn' - m'n)^2}$

For Skew lines the direction of the shortest distance would be perpendicular to both the lines.
If d = 0, the lines are coplanar

(iii) Shortest distance between two parallel lines $\vec{r}_1 = \vec{a}_1 + K\vec{b}$ and $\vec{r}_2 = \vec{a}_2 + K\vec{b}$, is given by

$d = \left| \dfrac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|} \right|$

Example # 29 : Find the shortest distance and the vector equation of the line of shortest distance between the lines given by $\vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + \lambda\left(3\hat{i} - \hat{j} + \hat{k}\right)$ and $\vec{r} = -3\hat{i} - 7\hat{j} + 6\hat{k} + \mu\left(-3\hat{i} + 2\hat{j} + 4\hat{k}\right)$

Solution : Equation of given lines in cartesian form is $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} = \lambda$ (say L_1)

and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4} = \mu$ (say L_2)

Let L on L_1 is $(3\lambda + 3, -\lambda + 8, \lambda + 3)$ and M on L_2 is $(-3\mu - 3, 2\mu - 7, 4\mu + 6)$

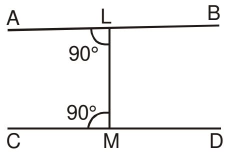
Direction ratios of LM are $3\lambda + 3\mu + 6, -\lambda - 2\mu + 15, \lambda - 4\mu - 3$.

Since LM $\perp$ L_1

$\therefore$ $3(3\lambda + 3\mu + 6) - 1(-\lambda - 2\mu + 15) + 1(\lambda - 4\mu - 3) = 0$ or, $11\lambda + 7\mu = 0$ ..... (1)

Again LM $\perp$ CD

$\therefore$ $-3(3\lambda + 3\mu + 6) + 2(-\lambda - 2\mu + 15) + 4(\lambda - 4\mu - 3) = 0$ or, $-7\lambda - 29\mu = 0$ ..... (2)



Solving (1) and (2), we get $\lambda = 0, \mu = 0 \Rightarrow$ $L \equiv (3, 8, 3), M \equiv (-3, -7, 6)$

Hence shortest distance LM = $\sqrt{(3+3)^2 + (8+7)^2 + (3-6)^2} = \sqrt{270} = 3\sqrt{30}$ units

Vector equation of LM is $\vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + t\left(6\hat{i} + 15\hat{j} - 3\hat{k}\right)$

Note : Cartesian equation of LM is $\frac{x-3}{6} = \frac{y-8}{15} = \frac{z-3}{-3}$.

Self practice problems:

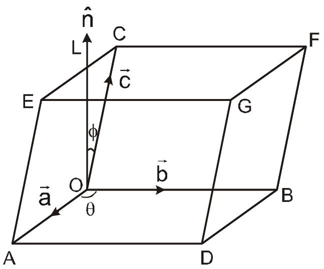
(33) Find the shortest distance between the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$. Find also its equation.

Answers : (33) $\frac{1}{\sqrt{6}}$, $6x - y = 10 - 3y = 6z - 25$

Scalar triple product (Box Product) (S.T.P.) :

(i) The scalar triple product of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is defined as: $\vec{a} \times \vec{b} \cdot \vec{c} = |\vec{a}|\,|\vec{b}|\,|\vec{c}|$, $\sin\theta \cdot \cos\phi$ where θ is the angle between $\vec{a}$, $\vec{b}$ **(i.e. $\vec{a} \wedge \vec{b} = \theta$)** and ϕ is the angle between $\vec{a} \times \vec{b}$ and $\vec{c}$ **$(\vec{a} \times \vec{b}) \wedge \vec{c} = \phi$)**. It is (i.e. $\vec{a} \times \vec{b} \cdot \vec{c}$) also written as $\left[\vec{a}\ \vec{b}\ \vec{c}\right]$ and spelled as box product.

(ii) Scalar triple product geometrically represents the volume of the parallelopiped whose three coterminous edges are



represented by $\vec{a}, \vec{b}$ and $\vec{c}$ i.e. $V = |[\vec{a}\ \vec{b}\ \vec{c}]|$

(iii) In a scalar triple product the position of dot and cross can be interchanged i.e.

$\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c} \Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

(iv) $\vec{a} \cdot (\vec{b} \times \vec{c}) = -\vec{a} \cdot (\vec{c} \times \vec{b})$ i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\ \vec{c}\ \vec{b}]$

(v) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, then $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

.

In general, if $\vec{a} = a_1\vec{\ell} + a_2\vec{m} + a_3\vec{n}$; $\vec{b} = b_1\vec{\ell} + b_2\vec{m} + b_3\vec{n}$ and $\vec{c} = c_1\vec{\ell} + c_2\vec{m} + c_3\vec{n}$

then $\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \left[\vec{\ell}\ \ \vec{m}\ \ \vec{n}\right]$, where $\vec{\ell}$, $\vec{m}$ and $\vec{n}$ are non-coplanar vectors.

(vi) If $\vec{a}, \vec{b}, \vec{c}$ are coplanar, then $\Rightarrow$ $[\vec{a}\ \vec{b}\ \vec{c}] = 0$.

(vii) If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar, then $[\vec{a}\ \vec{b}\ \vec{c}] > 0$ for right handed system and $[\vec{a}\ \vec{b}\ \vec{c}] < 0$ for left handed system.

(viii) $[\hat{i}\ \hat{j}\ \hat{k}] = 1$

(ix) $[K\vec{a}\ \vec{b}\ \vec{c}] = K\ [\vec{a}\ \vec{b}\ \vec{c}]$

(x) $[(\vec{a}+\vec{b})\vec{c}\,\vec{d}] = [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{b}\ \vec{c}\ \vec{d}]$

(xi) $\left[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}\right] = 0$ and $\left[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}\right] = 2\left[\vec{a}\ \vec{b}\ \vec{c}\right]$

(xii) $\left[\vec{a}\,\vec{b}\,\vec{c}\right]^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix}$

Volume of Parallelopiped/ Tetrahedron and their properties :

(a) The volume of the parallelopiped whose three coterminous edges are $\vec{a}$, $\vec{b}$ and $\vec{c}$ is $V = [\vec{a}\ \vec{b}\ \vec{c}]$

(b) The volume of the tetrahedron OABC with O as origin and the position vectors of A, B and C being $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively is given by $V = \frac{1}{6}\left|\left[\vec{a}\ \vec{b}\ \vec{c}\right]\right|$

(c) If the position vectors of the vertices of tetrahedron are $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$, then the position vector of its centroid is given by $\frac{1}{4}$ $(\vec{a}+\vec{b}+\vec{c}+\vec{d})$.

Note : that this is also the point of concurrency of the lines joining the vertices to the centroids of the opposite faces and is also called the centre of the tetrahedron. In case the tetrahedron is regular it is equidistant from the vertices and the four faces of the tetrahedron.

Example # 30 : The volume of the parallelopiped whose edges are represented by $-12\hat{i} + \lambda\hat{k}$, $3\hat{j} - k$, $2\hat{i} + \hat{j} - 15\hat{k}$ is 546, then find λ.

Solution : $V = \left|\begin{vmatrix} -12 & 0 & \lambda \\ 0 & 3 & -1 \\ 2 & 1 & -15 \end{vmatrix}\right|$ $\therefore$ $546 = |12 \times 44 - 6\lambda|$ $\therefore$ $\lambda = -3, 179$

Example # 31 : Find the volume of the tetrahedron whose four vertices have position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$.

Solution : Let four vertices be A, B, C, D with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ respectively.

$\therefore$ $\overrightarrow{DA} = (\vec{a} - \vec{d}) \Rightarrow \overrightarrow{DB} = (\vec{b} - \vec{d}) \Rightarrow \overrightarrow{DC} = (\vec{c} - \vec{d})$

Hence volume V = $\frac{1}{6}$ $[\vec{a} - \vec{d} \quad \vec{b} - \vec{d} \quad \vec{c} - \vec{d}]$

$= \frac{1}{6} (\vec{a} - \vec{d}).[(\vec{b} - \vec{d}) \times (\vec{c} - \vec{d})] = \frac{1}{6} (\vec{a} - \vec{d}).[\vec{b} \times \vec{c} - \vec{b} \times \vec{b} + \vec{c} \times \vec{d}]$

$= \frac{1}{6}\{[\vec{a}\ \vec{b}\ \vec{c}] - [\vec{a}\ \vec{b}\ \vec{d}] + [\vec{a}\ \vec{c}\ \vec{d}] - [\vec{d}\ \vec{b}\ \vec{c}]\} = \frac{1}{6}\{[\vec{a}\ \vec{b}\ \vec{c}] - [\vec{a}\ \vec{b}\ \vec{d}] + [\vec{a}\ \vec{c}\ \vec{d}] - [\vec{b}\ \vec{c}\ \vec{d}]\}$

Example # 32 : Prove that vectors $\vec{r_1}$ = ($\sec^2$ A, 1, 1) ; $\vec{r_2}$ = (1, $\sec^2$B, 1) ; $\vec{r_3}$ = (1, 1, $\sec^2$ C) are always non-coplanar vectors if A, B, C $\in$ (0, π).

Solution : Condition of coplanarity gives D = 0 $\Rightarrow$ $\begin{vmatrix} \sec^2 A & 1 & 1 \\ 1 & \sec^2 B & 1 \\ 1 & 1 & \sec^2 C \end{vmatrix} = 0$

$\Rightarrow$ $\sec^2A\,[\sec^2B\sec^2C - 1] - 1(\sec^2c - 1) + 1(1 - \sec^2B) = 0$

$\Rightarrow$ $(1 + \tan^2 A)(\tan^2 B + \tan^2 C + \tan^2 B \tan^2 C) - \tan^2 C - \tan^2 B = 0$

$\Rightarrow$ $\tan^2 B \tan^2 C + \tan^2 A \tan^2 B + \tan^2 C \tan^2 A + \tan^2 A \tan^2 B \tan^2 C = 0$

divide by $\tan^2 A \tan^2 B \tan^2 C$

$\cot^2A + \cot^2B + \cot^2C = -1$ it is a not possible

Example # 33 : If two pairs of opposite edges of a tetrahedron are mutually perpendicular, show that the third pair will also be mutually perpendicular.

Solution : Let OABC be the tetrahedron, where O is the origin and co-ordinates of A, B, C are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, x_3) respectively.

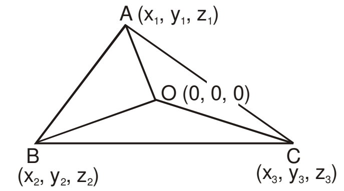
Let $OA \perp BC$ and $OB \perp CA$.

We have to prove that $OC \perp BA$.

Now, direction ratios of OA are x_1, y_1, z_1 and of BC are $(x_3 - x_2), (y_3 - y_2), (z_3 - z_2)$.

$\because$ $OA \perp BC$ and $OB \perp CA$

$\Rightarrow$ $x_1(x_3 - x_2) + y_1(y_3 - y_2) + z_1(z_3 - z_2) = 0$ and $x_2(x_1 - x_3) + y_2(y_1 - y_3) + z_2(z_1 - z_3) = 0$



Adding above two equations we get $x_3(x_1 - x_2) + y_3(y_1 - y_2) + z_3(z_1 - z_2) = 0$

$\therefore$ $OC \perp BA$ ($\because$ direction ratios of OC are x_3, y_3, z_3 and that of BA are $(x_1 - x_2), (y_1 - y_2), (z_1 - z_2)$)

Self practice problems :

(34) Show that $\vec{a} \,.\, (\vec{b} + \vec{c}) \times (\vec{a} + \vec{b} + \vec{c}) = 0$

(35) One vertex of a parallelopiped is at the point A (1, –1, –2) in the rectangular cartesian co- ordinate. If three adjacent vertices are at B(–1, 0, 2), C(2, –2, 3) and D(4, 2, 1), then find the volume of the parallelopiped.

(36) Show that the vector $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar if and only if $\vec{b}+\vec{c}$, $\vec{c}+\vec{a}$, $\vec{a}+\vec{b}$ are coplanar.

(37) Show that $\{(\vec{a}+\vec{b}+\vec{c}) \times (\vec{c}-\vec{b})\} . \vec{a} = 2\left[\vec{a}\ \vec{b}\ \vec{c}\right]$.

(38) Find the value of m such that the vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}+2\hat{j}-3\hat{k}$ and $3\hat{i}+m\hat{j}+5\hat{k}$ are coplanar.

(39) Find the value of λ for which the four points with position vectors $-\hat{j}-\hat{k}$, $4\hat{i}+5\hat{j}+\lambda\hat{k}$, $3\hat{i}+9\hat{j}+4\hat{k}$, and $-4\hat{i}+4\hat{j}+4\hat{k}$ are coplanar.

Answer : (35) 72 (38) – 4 (39) $\lambda = 1$

Vector triple product :

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be any three vectors, then the expression $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector & is called a vector triple product. This vector is perpendicular to $\vec{a}$ and lies in plane containing vectors $\vec{b}$ and $\vec{c}$

- $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$
- $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$
- In general $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$

Example # 34 : $[\vec{a} \times (3\vec{b}+2\vec{c})\ \ \vec{b} \times (\vec{c}-2\vec{a})\ \ \ 2\vec{c} \times (\vec{a}-3\vec{b})] =$

Solution : Let $\vec{b} \times \vec{c} = \vec{p}$, $\vec{c} \times \vec{a} = \vec{q}$, $\vec{a} \times \vec{b} = \vec{r}$

$\therefore$ $[\vec{p}\ \vec{q}\ \vec{r}] = [\vec{a}\ \vec{b}\ \vec{c}]^2$ ....(i)

$\vec{a} \times (3\vec{b}+2\vec{c}) = 3\vec{r} - 2q$ etc.

$\therefore$ $E = [3\vec{r} - 2\vec{q}\ \ \vec{p} + 2\vec{r}\ ,\ 2\vec{q} + 6\vec{p}]$

$$= [0\vec{p} - 2\vec{q} + 3\vec{r}\ ,\ \vec{p} + 0\vec{q} + 2\vec{r}\ ,\ 6\vec{p} + 2\vec{q} + 0\vec{r}] = \begin{vmatrix} 0 & -2 & 3 \\ 1 & 0 & 2 \\ 6 & 2 & 0 \end{vmatrix} [\vec{a}\ \vec{b}\ \vec{c}]^2 = -18\,[\vec{a}\ \vec{b}\ \vec{c}]^2$$

Example # 35 : If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar unit vectors such that $(\vec{a} \times \vec{b}) \times \vec{c} = \left(\dfrac{\sqrt{3}\vec{b}+\vec{a}}{2}\right)$. Then find angles which makes $\vec{c}$ with $\vec{a}$ & $\vec{b}$ ($\vec{a}$ and $\vec{b}$ are non-collinear)

Solution : $(\vec{a} \times \vec{b}) \times \vec{c} = \left(\dfrac{\sqrt{3}\vec{b}+\vec{a}}{2}\right) \Rightarrow (\vec{a}.\vec{c}.),\vec{b} - (\vec{b}.\vec{c})\vec{a} = \dfrac{\sqrt{3}\vec{b}+\vec{a}}{2}$

$\vec{a}.\vec{c} = \dfrac{\sqrt{3}}{2}$, and $\vec{b}.\vec{c} = -\dfrac{1}{2}$.

$\cos\theta = \dfrac{\sqrt{3}}{2}$ and $\cos\phi = -\dfrac{1}{2}$.

$\theta = \dfrac{\pi}{6}$ and $\phi = \dfrac{2\pi}{3}$

Example # 36 : Prove that $\vec{a}\times\{\vec{b}\times(\vec{c}\times\vec{d})\} = (\vec{b}\ .\ \vec{d})(\vec{a}\times\vec{c}) - (\vec{b}.\ \vec{c})\ (\vec{a}\times\vec{d})$

Solution : We have, $\vec{a}\times\{\vec{b}\times(\vec{c}\times\vec{d})\} = \vec{a}\times\{(\vec{b}\ .\ \vec{d})\ \vec{c} - (\vec{b}\ .\ \vec{c})\ \vec{d}\}$

$= \vec{a}\times\{(\vec{b}\ .\ \vec{d})\ \vec{c}\} - \vec{a}\times\{(\vec{b}\ .\ \vec{c})\ \vec{d}\}$ [by dist. law]

$= (\vec{b}\ .\ \vec{d})\ (\vec{a}\times\vec{c}) - (\vec{b}\ .\ \vec{c})\ (\vec{a}\times\vec{d})$.

Self Practice Problems :

(40) Prove that $\vec{a}\times\{\vec{a}\times(\vec{a}\times\vec{b})\} = (\vec{a}\ .\ \vec{a})\ (\vec{b}\times\vec{a})$.

(41) Let $\vec{b}$ and $\vec{c}$ be noncollinear vectors. If $\vec{a}$ is a vector such that $\vec{a}\,.(\vec{b}+\vec{c}) = 4$ and $\vec{a}\times(\vec{b}\times\vec{c}) = (x^2 - 2x + 6)\ \vec{b} + \vec{c}$ siny, then find x and y.

(42) Find a unit vector coplanar with $\hat{i}+\hat{j}+2\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$ and perpendicular to $\hat{i}+\hat{j}+\hat{k}$ is

Answer : (41) $x = 1$ & $y = (4n + 1)\ \pi/2$, $n \in I$ (42) $\pm\left(\dfrac{\hat{j}-\hat{k}}{\sqrt{2}}\right)$

Linear combinations :

Given a finite set of vectors $\vec{a},\vec{b},\vec{c},......$, then the vector $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} + ...$ is called a linear combination of $\vec{a},\vec{b},\vec{c},......$ for any x, y, z..... $\in$ R. We have the following results :

(a) If $\vec{a},\vec{b}$ are non zero, non–collinear vectors, then $x\vec{a}+y\vec{b} = x'\vec{a}+y'\vec{b} \Rightarrow x = x'$, $y = y'$

(b) **Fundamental Theorem in plane :** Let $\vec{a},\vec{b}$ be non zero, non collinear vectors, then any vector $\vec{r}$ coplanar with $\vec{a},\vec{b}$ can be expressed uniquely as a linear combination of $\vec{a}$ and $\vec{b}$
i.e. there exist some unique x, y $\in$ R such that $x\vec{a}+y\vec{b}=\vec{r}$.

(c) If $\vec{a},\vec{b},\vec{c}$ are non–zero, non–coplanar vectors, then
$x\vec{a}+y\vec{b}+z\vec{c}=x'\vec{a}+y'\vec{b}+z'\vec{c} \Rightarrow x = x'$, $y = y'$, $z = z'$

(d) **Fundamental theorem in space:** Let $\vec{a},\vec{b},\vec{c}$ be non–zero, non–coplanar vectors in space. Then any vector $\vec{r}$ can be uniquely expressed as a linear combination of $\vec{a},\vec{b},\vec{c}$ i.e. there exist some unique x,y, z $\in$ R such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{r}$.

(e) If $\vec{x}_1,\vec{x}_2,......,\ \vec{x}_n$ are n non zero vectors and $k_1,k_2,.....,k_n$ are n scalars and if the linear combination $k_1\vec{x}_1 + k_2\vec{x}_2 + ... + k_n\vec{x}_n = \vec{0} \Rightarrow k_1 = 0,\ k_2 = 0\ ,\ .....,\ k_n = 0$, then we say that vectors $\vec{x}_1,\ \vec{x}_2,\ ......,\vec{x}_n$ are **linearly independent vectors**.

(f) If $k_1\vec{x}_1 + k_2\vec{x}_2 + k_3\vec{x}_3 ...... + k_r\vec{x}_r + ... + k_n\vec{x}_n = \vec{0}$ and if there exists at least one $k_r \neq 0$, then $\vec{x}_1,\ \vec{x}_2,\ ......,\ \vec{x}_n$ are said to be **linearly dependent vectors.**
If $k_r \neq 0$ then $\vec{x}_r$ is expressed as a linear combination of vectors $\vec{x}_1,\ \vec{x}_2,........,\ \vec{x}_{r-1},\ \vec{x}_{r+1},\ ....,\ \vec{x}_n$

Note :

☞ In general, in 3 dimensional space every set of four vectors is a linearly dependent system.

☞ $\hat{i}\ ,\hat{j}\ ,\hat{k}$ are **Linearly Independent** set of vectors. For $K_1\hat{i} + K_2\hat{j} + K_3\hat{k} = \vec{0} \Rightarrow K_1 = K_2 = K_3 = 0$

☞ Two vectors $\vec{a}$ and $\vec{b}$ are linearly dependent $\Rightarrow$ $\vec{a}$ is parallel to $\vec{b}$ i.e. $\vec{a}\times\vec{b} = \vec{0} \Rightarrow$ linear dependence of $\vec{a}$ and $\vec{b}$. Conversely if $\vec{a}\times\vec{b} \neq \vec{0}$ then $\vec{a}$ and $\vec{b}$ are linearly independent.

☞ If three vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly dependent, then they are coplanar i.e. $[\vec{a}\ \ \vec{b}\ \ \vec{c}] = 0$. Conversely if $[\vec{a}\ \ \vec{b}\ \ \vec{c}] \neq 0$ then the vectors are linearly independent.

Example # 37 : If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three non-coplanar vectors, solve the vector equation $\vec{r}.\vec{a} = \vec{r}.\vec{b} = \vec{r}.\vec{c} = 1$

Solution : since $\vec{a}$, $\vec{b}$, $\vec{c}$ are three non-coplanar vectors therefore $\vec{a} \times \vec{b}$, $\vec{b} \times \vec{c}$ & $\vec{c} \times \vec{a}$ are also non-coplanar vectors

Let $\vec{r} = x(\vec{a} \times \vec{b}) + y(\vec{b} \times \vec{c}) + z(\vec{c} \times \vec{a})$.

Then, $\vec{r}.\vec{a} = 1 \Rightarrow \quad 1 = y[(\vec{b} \times \vec{c})\ \vec{a}.]$

$y = \dfrac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}$, similarly $x = z = \dfrac{1}{[\vec{a}\ \vec{b}\ \vec{c}]} \Rightarrow \vec{r} = \dfrac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}\ ((\vec{a} \times \vec{b}) + (\vec{b} \times \vec{c}) + (\vec{c} \times \vec{a}))$

Example # 38 : Given that position vectors of points A, B, C are respectively

$\vec{a} - 2\vec{b} + 3\vec{c}$, $2\vec{a} + 3\vec{b} - 4\vec{c}$, $-7\vec{b} + 10\vec{c}$ then prove that vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are linearly dependent.

Solution : Let A, B, C be the given points and O be the point of reference then

$\overrightarrow{OA} = \vec{a} - 2\vec{b} + 3\vec{c}$, $\overrightarrow{OB} = 2\vec{a} + 3\vec{b} - 4\vec{c}$ and $\overrightarrow{OC} = -7\vec{b} + 10\vec{c}$

Now $\overrightarrow{AB}$ = p.v. of B – p.v. of A

$= \overrightarrow{OB} - \overrightarrow{OA} = (\vec{a} + 5\vec{b} - 7\vec{c})$ and $\overrightarrow{AC}$ = p.v. of C – p.v of A

$= \overrightarrow{OC} - \overrightarrow{OA} = -(\vec{a} + 5\vec{b} - 7\vec{c}) = -\overrightarrow{AB}$

$\therefore \quad \overrightarrow{AC} = \lambda\ \overrightarrow{AB}$

where $\lambda = -1$. Hence $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are linearly dependent.

Example # 39 : Prove that the vectors $5\vec{a} + 6\vec{b} + 7\vec{c}$, $7\vec{a} - 8\vec{b} + 9\vec{c}$ and $3\vec{a} + 20\vec{b} + 5\vec{c}$ are linearly dependent, where $\vec{a}, \vec{b}, \vec{c}$ being linearly independent vectors.

Solution : We know that if these vectors are linearly dependent , then we can express one of them as a linear combination of the other two.

Now let us assume that the given vector are coplanar, then we can write

$5\vec{a} + 6\vec{b} + 7\vec{c} = \ell(7\vec{a} - 8\vec{b} + 9\vec{c}) + m(3\vec{a} + 20\vec{b} + 5\vec{c})$ where ℓ, m are scalars

Comparing the coefficients of $\vec{a}$, $\vec{b}$ and $\vec{c}$ on both sides of the equation

$5 = 7\ell + 3m, \quad 6 = -8\ell + 20m, \ 7 = 9\ell + 5m$

$\Rightarrow \ \ell = \dfrac{1}{2} = m$. Hence the given vectors are linearly dependent.

Self Practice Problems :

(43) Given that $\vec{x} + \dfrac{1}{\vec{p}^2}(\vec{p}\ .\ \vec{x})\ \vec{p} = \vec{q}$, show that $\vec{p}\ .\ \vec{x} = \dfrac{1}{2}\vec{p}\ .\ \vec{q}$ and find $\vec{x}$ in terms of $\vec{p}$ and $\vec{q}$.

(44) If $\vec{x}\ .\ \vec{a} = 0$, $\vec{x}\ .\ \vec{b} = 0$ and $\vec{x}\ .\ \vec{c} = 0$ for some non-zero vector $\vec{x}$, then show that $[\vec{a}\ \ \vec{b}\ \ \vec{c}] = 0$

(45) Prove that $\vec{r} = \dfrac{(\vec{r}\ .\ \vec{a})\ (\vec{b} \times \vec{c})}{[\ \vec{a}\ \ \vec{b}\ \ \vec{c}\ ]} + \dfrac{(\vec{r}\ .\ \vec{b})\ (\vec{c} \times \vec{a})}{[\ \vec{a}\ \ \vec{b}\ \ \vec{c}\ ]} + \dfrac{(\vec{r}\ .\ \vec{c})\ (\vec{a} \times \vec{b})}{[\ \vec{a}\ \ \vec{b}\ \ \vec{c}\ ]}$

where $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors

(46) Does there exist scalars u, v, w such that $u\vec{e}_1 + v\vec{e}_2 + w\vec{e}_3 = \hat{i}$ where $\vec{e}_1 = \hat{k}$, $\vec{e}_2 = \hat{j} + \hat{k}$, $\vec{e}_3 = -\hat{j} + 2\hat{k}$?

(47) If $\vec{a}$ and $\vec{b}$ are non-collinear vectors and $\vec{A} = (x + 4y)\,\vec{a} + (2x + y + 1)\,\vec{b}$ and $\vec{B} = (y - 2x + 2)\,\vec{a} + (2x - 3y - 1)\,\vec{b}$, find x and y such that $3\vec{A} = 2\vec{B}$.

(48) If vectors $\vec{a}, \vec{b}, \vec{c}$ be linearly independent, then show that

(i) $\vec{a} - 2\vec{b} + 3\vec{c}$, $-2\vec{a} + 3\vec{b} - 4\vec{c}$, $-\vec{b} + 2\vec{c}$ are linearly dependent

(ii) $\vec{a} - 3\vec{b} + 2\vec{c}$, $-2\vec{a} - 4\vec{b} - \vec{c}$, $3a + 2\vec{b} - \vec{c}$ are linearly independent.

(49) Prove that a vector $\vec{r}$ in space can be expressed linearly in terms of three non-coplanar, non-zero vectors $\vec{a}, \vec{b}, \vec{c}$ in the form

$$\vec{r} = \frac{[\vec{r}\ \vec{b}\ \vec{c}]\ \vec{a} + [\vec{r}\ \vec{c}\ \vec{a}]\ \vec{b} + [\vec{r}\ \vec{a}\ \vec{b}]\ \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

Answers : (43) $\vec{x} = \vec{q} - \left(\frac{\vec{p}\ .\ \vec{q}}{2\,|\vec{p}|^2}\right)\vec{p}$ (46) No (47) $x = 2, y = -1$

Test of collinearity :

Three points A,B,C with position vectors $\vec{a}, \vec{b}, \vec{c}$ respectively are collinear, if & only if there exist scalars x, y, z not all zero simultaneously such that $x\vec{a} + y\vec{b} + z\vec{c} = 0 = \vec{0}$, where $x + y + z = 0$.

Test of coplanarity :

Four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively are coplanar if and only if there exist scalars x, y, z, w not all zero simultaneously such that $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = \vec{0}$, where $x + y + z + w = 0$.

Example # 40 : Prove that four points $2\vec{a} + 3\vec{b} - \vec{c}$, $\vec{a} - 2\vec{b} + 3\vec{c}$, $3\vec{a} + 4\vec{b} - 2\vec{c}$ and $\vec{a} - 6\vec{b} + 6\vec{c}$ are coplanar.

Solution : Let the given four points be P, Q, R and S respectively. These points are coplanar if the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{PS}$ are coplanar. These vectors are coplanar iff one of them can be expressed as a linear combination of other two. So let $\overrightarrow{PQ} = x\overrightarrow{PR} + y\,\overrightarrow{PS}$

$\Rightarrow \quad -\vec{a} - 5\vec{b} + 4\vec{c} = x\left(\vec{a} + \vec{b} - \vec{c}\right) + y\left(-\vec{a} - 9\vec{b} + 7\vec{c}\right)$

$\Rightarrow \quad -\vec{a} - 5\vec{b} + 4\vec{c} = (x - y)\,\vec{a} + (x - 9y)\,\vec{b} + (-x + 7y)\,\vec{c}$

$\Rightarrow \quad x - y = -1,\ x - 9y = -5,\ -x + 7y = 4$ [Equating coeff. of $\vec{a}, \vec{b}, \vec{c}$ on both sides]

Solving the first two equations of these three equations, we get $x = -\frac{1}{2}$, $y = \frac{1}{2}$.

These values also satisfy the third equation. Hence the given four points are coplanar.

Self Practice Problems :

(50) If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are any four vectors in 3-dimensional space with the same initial point and such that $3\vec{a} - 2\vec{b} + \vec{c} - 2\vec{d} = \vec{0}$, show that the terminal A, B, C, D of these vectors are coplanar. Find the point (P) at which AC and BD meet. Also find the ratio in which P divides AC and BD.

Answers : (50) $\vec{p} = \frac{3\vec{a} + \vec{c}}{4}$ divides AC in 1 : 3 and BD in 1 : 1 ratio

A PLANE

If line joining any two points on a surface lies completely on it then the surface is a plane.

OR

If line joining any two points on a surface is perpendicular to some fixed straight line. Then this surface is called a plane. This fixed line is called the normal to the plane.

Equation of a plane :

(i) **Vector form :** The equation $(\vec{r} - \vec{r}_0) \cdot \vec{n} = 0$ represents a plane containing the point with position vector is a vector normal to the plane.
The above equation can also be written as $\vec{r} \cdot \vec{n} = d$, where d = $\vec{r}_0 \cdot \vec{n}$

(ii) **Cartesian form :** The equation of a plane passing through the point (x_1, y_1, z_1) is given by $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ where a, b, c are the direction ratios of the normal to the plane.

(iii) **Normal form :** Vector equation of a plane normal to unit vector and at a distance d from the origin is $\vec{r} \cdot \vec{n} = d$. Normal form of the equation of a plane is $\ell x + my + nz = p$, where, ℓ,m, n are the direction cosines of the normal to the plane and p is the distance of the plane from the origin.

(iv) **General form :** $ax + by + cz + d = 0$ is the equation of a plane, where a, b, c are the direction ratios of the normal to the plane.

(v) **Plane through three points :** The equation of the plane through three non–collinear points $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ is $\begin{vmatrix} x - x_3 & y - y_3 & z - z_3 \\ x_1 - x_3 & y_1 - y_3 & z_1 - z_3 \\ x_2 - x_3 & y_2 - y_3 & z_2 - z_3 \end{vmatrix} = 0$

(vi) **Intercept Form :** The equation of a plane cutting intercept a, b, c on the axes is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Note :

☞ Equation of yz–plane, xz–plane and xy–plane is x = 0, y = 0 and z = 0

☞ **Transformation of the equation of a plane to the normal form:** To reduce any equation $ax + by + cz - d = 0$ to the normal form, first write the constant term on the right hand side and make it positive, then divide each term by $\sqrt{a^2 + b^2 + c^2}$, where a, b, c are coefficients of x, y and z respectively e.g.

$$\frac{ax}{\pm\sqrt{a^2 + b^2 + c^2}} + \frac{by}{\pm\sqrt{a^2 + b^2 + c^2}} + \frac{cz}{\pm\sqrt{a^2 + b^2 + c^2}} = \frac{d}{\pm\sqrt{a^2 + b^2 + c^2}}$$

Where (+) sign is to be taken if d > 0 and (–) sign is to be taken if d < 0.

☞ A plane $ax + by + cz + d = 0$ divides the line segment joining (x_1, y_1, z_1) and (x_2, y_2, z_2). in the ratio $\left(-\frac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d}\right)$

☞ Coplanarity of four points

The points $A(x_1\, y_1\, z_1)$, $B(x_2\, y_2\, z_2)$ $C(x_3\, y_3\, z_3)$ and $D(x_4\, y_4\, z_4)$ are coplanar then

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \\ x_4-x_1 & y_4-y_1 & z_4-z_1 \end{vmatrix} = 0$$

Example # 41 : Find the equation of the plane upon which the length of normal from origin is 10 and direction ratios of this normal are 3, 2, 6.

Solution : If p be the length of perpendicular from origin to the plane and ℓ, m, n be the direction cosines of this normal, then its equation is

$\ell x + my + nz = 10$ ..... (1)

Direction ratios of normal to the plane are 3, 2, 6

$\therefore$ Direction cosines of normal to the required plane are $\ell = \frac{3}{7}$, $m = \frac{2}{7}$, $n = \frac{6}{7}$

Equation of required plane is $\frac{3}{7}x + \frac{2}{7}y + \frac{6}{7}z = 10$ or, $3x + 2y + 6z = 70$

Example # 42 : Find the plane through the points (2,–3,3), (–5, 2, 0), (1, – 7, 1)

Solution : $$\begin{vmatrix} x-2 & y+3 & z-3 \\ -5-2 & 2+3 & 0-3 \\ 1-2 & -7+3 & 1-3 \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} x-2 & y+3 & z-3 \\ -7 & 5 & -3 \\ -1 & -4 & -2 \end{vmatrix} = 0 \Rightarrow 2x + y - 3z + 8 = 0$$

Example # 43 : If P be any point on the plane $\ell x + my + nz = p$ and Q be a point on the line OP such that $OP \,.\, OQ = p^2$, show that the locus of the point Q is $p(\ell x + my + nz) = x^2 + y^2 + z^2$.

Solution : Let $P \equiv (\alpha, \beta, \gamma)$, $Q \equiv (x_1, y_1, z_1)$

Direction ratios of OP are α, β, γ and direction ratios of OQ are x_1, y_1, z_1.

Since O, Q, P are collinear, we have $\frac{\alpha}{x_1} = \frac{\beta}{y_1} = \frac{\gamma}{z_1} = k$ (say) ..... (1)

As P (α, β, γ) lies on the plane $\ell x + my + nz = p$,

$\ell\alpha + m\beta + n\gamma = p$ or $k(\ell x_1 + my_1 + nz_1) = p$ ..... (2)

Given $OP \,.\, OQ = p^2$ $\Rightarrow$ $\sqrt{\alpha^2+\beta^2+\gamma^2}\ \sqrt{x_1^2+y_1^2+z_1^2} = p^2$

or, $\sqrt{k^2(x_1^2+y_1^2+z_1^2)}\ \sqrt{x_1^2+y_1^2+z_1^2} = p^2$ or, $k\ (x_1^2+y_1^2+z_1^2) = p^2$ ....(3)

On dividing (2) by (3), we get $\frac{\ell x_1 + my_1 + nz_1}{x_1^2+y_1^2+z_1^2} = \frac{1}{p}$ or, $p\,(\ell x_1 + my_1 + nz_1) = x_1^2+y_1^2+z_1^2$

Hence the locus of point Q is $p\,(\ell x + my + nz) = x^2 + y^2 + z^2$.

Example # 44 : A moving plane passes through a fixed point (α,β,γ) and cuts the coordinate axes A, B, C . Find the locus of the centroid of the tetrahedron OABC.

Solution : Let the plane be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, 0 (0,0,0) , A (a, 0,0), B (0, b, 0)

C (0,0,c) . Centroid of OABC is $\left(\frac{a}{4}, \frac{b}{4}, \frac{c}{4}\right)$

The plane passes through (α,β,γ) $\therefore$ $\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 1$ ......(i)

Centroid , $x = \frac{a}{4}$, $y = \frac{b}{4}$, $z = \frac{c}{4}$ or $a = 4x$, $b = 4y$, $c = 4z$

Now (1) gives the locus of G as $\frac{\alpha}{x} + \frac{\beta}{y} + \frac{\gamma}{z} = 4$

Self practice problems :

(51) Check whether given points are coplanar if yes find the equation of plane containing them
$A \equiv (0, -1, -1)$, $B \equiv (4, 5, 1)$, $C \equiv (3, 9, 4)$, $D \equiv (-4, 4, 4)$

(52) Find the plane passing through point (3, 2, 1) and perpendicular to the line joining the points (2, 4, 3) and (3, –1, 5).

(53) Find the equation of plane passing through the point (2, 4, 6) and making equal intercepts on the coordinate axes.

(54) Find the equation of plane passing through (1, 2, –3) and (2, 3, 3) and perpendicular to the plane $2x + y - 3z + 4 = 0$.

(55) Find the equation of the plane parallel to $2\hat{i} + \hat{j} - \hat{k}$ and $\hat{i} - 2\hat{j} - 3\hat{k}$ and passing through (2, 1, 3).

(56) Find the equation of the plane passing through the point (1, 1, – 1) and perpendicular to the planes $x + 2y + 3z - 7 = 0$ and $2x - 3y + 4z = 0$.

Answers :
(51) yes, $5x - 7y + 11z + 4 = 0$
(52) $x - 5y + 2z + 5 = 0$
(53) $x + y + z = 12$
(54) $9x - 15y + z + 24 = 0$
(55) $x - y + z = 4$
(56) $17x + 2y - 7z = 26$

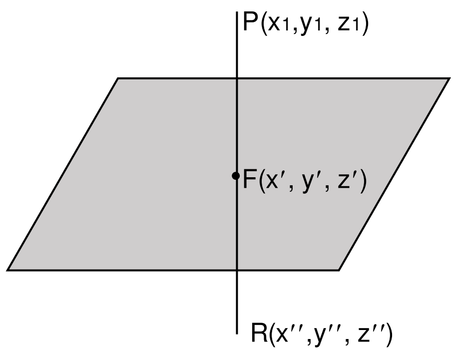
Position of point with respect to plane :

A plane divides the three dimensional space in two equal parts. Two points A $(x_1\ y_1\ z_1)$ and B $(x_2\ y_2\ z_2)$ are on the same side of the plane $ax + by + cz + d = 0$ if $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ are both positive or both negative and are opposite side of plane if both of these values are in opposite sign.

Example # 45 : Show that the points (1, 2, 3) and (2, – 1, 4) lie on opposite sides of the plane $x + 4y + z - 3 = 0$.

Solution : Since the numbers $1 + 4 \times 2 + 3 - 3 = 9$ and $2 - 4 + 4 - 3 = -1$ are of opposite sign, then points are on opposite sides of the plane.

A plane & a point



Let $P \equiv ax + by + cz + d = 0$ is a given plane and $P(x_1, y_1, z_1)$ is given point as shown in figure.

Let $F(x', y', z')$ be the foot of the point $P(x_1, y_1, z_1)$ with respect to the plane P.

And $R(x'', y'', z'')$ be the reflection of point $P(x_1, y_1, z_1)$ with respect to the plane P.

(i) Distance of the point (x_1, y_1, z_1) from the plane ax + by + cz+ d = 0 is given by $\left|\dfrac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$.

(ii) The length of the perpendicular from a point having position vector $\vec{a}$ to plane $\vec{r}.\vec{n}$ = d is $p = \dfrac{|\vec{a}\ .\ \vec{n}-d|}{|\vec{n}|}$.

(iii) The coordinates of the foot (F) of perpendicular from the point (x_1, y_1, z_1) to the plane ax + by + cz + d = 0 are $\dfrac{x-x_1}{a}=\dfrac{y-y_1}{b}=\dfrac{z-z_1}{c}=-\dfrac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$

(iv) The coordinates of the Image (R) of point (x_1, y_1, z_1) to the plane ax + by + cz + d = 0 are $\dfrac{x-x_1}{a}=\dfrac{y-y_1}{b}=\dfrac{z-z_1}{c}=-\dfrac{2(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$

Example # 46 : Find the image of the point P (3, 5, 7) in the plane 2x + y + z = 0.

Solution : Given plane is 2x + y + z = 0 ..... (1)

Direction ratios of normal to plane (1) are 2, 1, 1

Let Q be the image of point P in plane (1). Let PQ meet plane (1) in R then PQ $\perp$ plane (1)

Let R $\equiv$ (2r + 3, r + 5, r + 7)

Since R lies on plane (1)

$\therefore$ 2(2r + 3) + r + 5 + r + 7 = 0 or, 6r + 18 = 0 $\therefore$ r = – 3

$\therefore$ R $\equiv$ (– 3, 2, 4)

Let Q $\equiv$ (α, β, γ)

Since R is the middle point of PQ

$\therefore$ $-3 = \dfrac{\alpha+3}{2} \Rightarrow \alpha = -9$ and $2 = \dfrac{\beta+5}{2} \Rightarrow \beta = -1$ and $4 = \dfrac{\gamma+7}{2} \Rightarrow \gamma = 1$

$\therefore$ Q = (– 9, – 1, 1).

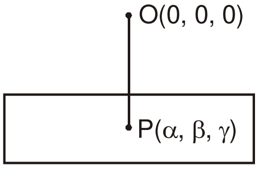
Example # 47 : A plane passes through a fixed point (a, b, c). Show that the locus of the foot of perpendicular to it from the origin is the sphere $x^2 + y^2 + z^2 - ax - by - cz = 0$

Solution : Let the equation of the variable plane be $\ell x + my + nz + d = 0$ ..... (1)

Plane passes through the fixed point (a, b, c) $\therefore$ $\ell a + mb + nc + d = 0$ ..... (2)

Let P (α, β, γ) be the foot of perpendicular from origin to plane (1).

Direction ratios of OP are



$\alpha - 0, \beta - 0, \gamma - 0$ i.e. α, β, γ

From equation (1), it is clear that the direction ratios of normal to the plane i.e. OP are ℓ, m, n;

α, β, γ and ℓ, m, n are the direction ratios of the same line OP

$\therefore$ $\dfrac{\alpha}{\ell} = \dfrac{\beta}{m} = \dfrac{\gamma}{n} = \dfrac{1}{k}$ (say) $\therefore$ $\ell = k\alpha$, $m = k\beta$, $n = k\gamma$ ..... (3)

Putting the values of ℓ, m, n in equation (2), we get $ka\alpha + kb\beta + kc\gamma + d = 0$ ..... (4)

Since α, β, γ lies in plane (1) $\therefore$ $\ell\alpha + m\beta + n\gamma + d = 0$ ..... (5)

Putting the values of ℓ, m, n from (3) in (5), we get $k\alpha^2 + k\beta^2 + k\gamma^2 + d = 0$ ..... (6)

or $k\alpha^2 + k\beta^2 + k\gamma^2 - ka\alpha - kb\beta - kc\gamma = 0$ [putting the value of d from (4) in (6)]

or $\alpha^2 + \beta^2 + \gamma^2 - a\alpha - b\beta - c\gamma = 0$

Therefore, locus of foot of perpendicular P (α, β, γ) is $x^2 + y^2 + z^2 - ax - by - cz = 0$ ..... (7)

Self practice problems :

(57) Find the intercepts of the plane 3x + 4y – 7z = 84 on the axes. Also find the length of perpendicular from origin to this plane and direction cosines of this normal.

(58) Find : (i) perpendicular distance (ii) foot of perpendicular
(iii) image of (1, 1, 1) in the plane 3x + 4y – 12z + 13 = 0

Answers : (57) a = 28, b = 21, c = – 12, p = $\frac{84}{\sqrt{74}}$; $\frac{3}{\sqrt{74}}$, $\frac{4}{\sqrt{74}}$, $\frac{-7}{\sqrt{74}}$

(58) (i) $\frac{\sqrt{17}}{2}$ (ii) (–1, 1/2, 1) (iii) (–3, 0,1)

Angle between two planes :

(i) Consider two planes ax + by + cz + d = 0 and a′x + b′y + c′z + d′ = 0. Angle between these planes is the angle between their normals. Since direction ratios of their normals are (a, b, c) and (a′, b′, c′) respectively, hence θ, the angle between them, is given by

$$\cos\theta = \frac{aa'+bb'+cc'}{\sqrt{a^2+b^2+c^2}\ \sqrt{a'^2+b'^2+c'^2}}$$

Planes are perpendicular if aa′ + bb′ + cc′ = 0 and planes are parallel if $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$

(ii) The angle θ between the planes $\vec{r}\cdot\vec{n}_1 = d_1$ and $\vec{r}\cdot\vec{n}_2 = d_2$ is given by, $\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\ |\vec{n}_2|}$

Planes are perpendicular if $\vec{n}_1 \cdot \vec{n}_2 = 0$ & planes are parallel if $\vec{n}_1 = \lambda\vec{n}_2$.

Distance between parallel planes :

Distance between two parallel planes ax + by + cz + d_1 = 0 and ax + by + cz + d_2 = 0 is $\frac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}}$

Example # 48 : Find the distance between the parallel planes 2x – y + 2z + 3 = 0 and 4x – 2y + 4z + 5 = 0

Solution : Given planes are 2x – y + 2z + 3 = 0 and 2x – y + 2z + 5/2 = 0

Required distance between planes = $\frac{|3-5/2|}{\sqrt{(2)^2+(-1)^2+(2)^2}} = \frac{1}{6}$

Angle bisectors

(i) The equations of the planes bisecting the angle between two given planes
$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}} \qquad \text{......(1)}$$

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = -\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}} \qquad \text{........(2)}$$

(ii) If $a_1\alpha + b_1\beta + c_1\gamma + d_1$ and $a_2\alpha + b_2\beta + c_2\gamma + d_2$ are of same/opposite sign then (1)/(2) gives equation of angle bisector of region containing point (α,β,γ)

(iii) If $a_1a_2 + b_1b_2 + c_1c_2 > 0$, then equation (1)/(2) gives obtuse/acute angle bisector and if $a_1a_2 + b_1b_2 + c_1c_2 < 0$, then equation (1)/(2) gives acute/obtuse angle bisector.

Family of planes

(i) Any plane passing through the line of intersection of non–parallel planes or equation of the plane through the given line in non symmetric form.

$a_1x + b_1y + c_1z + d_1 = 0$ & $a_2x + b_2y + c_2z + d_2 = 0$ is

$a_1x + b_1y + c_1z + d_1 + \lambda\,(a_2x + b_2y + c_2z + d_2) = 0$, where $\lambda \in R$

(ii) The equation of plane passing through the intersection of the planes $\vec{r}\,.\,\vec{n}_1 = d_1$ & $\vec{r}\,.\vec{n}_2 = d_2$ is. $\vec{r}\,(n_1 + \lambda\,\vec{n}_2) = d_1 + \lambda d_2$ where λ is arbitrary scalar

Example # 49 : Find the equation of the plane through the line of intersection of the planes $x + 2y + 3z + 2 = 0$, $2x + 3y - z + 3 = 0$ and perpendicular to the plane $x + y + z = 0$

Solution : The plane is $x + 2y + 3z + 2 + \lambda\,(2x + 3y - z + 3) = 0$

or $(1 + 2\lambda)\,x + (2 + 3\lambda)\,y + (3 - \lambda)\,z + 2 + 3\lambda = 0$

It is perpendicular to $x + y + z = 0$

$\therefore\ 1 + 2\lambda + 2 + 3\lambda + 3 - \lambda = 0$ or $2\lambda + 3 = 0 \Rightarrow \lambda = -\dfrac{3}{2}$

Substituting we get $4x + 5y - 9z + 5 = 0$

Example # 50 : Find the equation of the plane through the point (1, 1, 1) which passes through the line of intersection of the planes $x + y + z = 6$ and $2x + 3y + 4z + 5 = 0$.

Solution : Given planes are $x + y + z - 6 = 0$ ..... (1)

and $2x + 3y + 4z + 5 = 0$ ..... (2)

Given point is P (1, 1, 1).

Equation of any plane through the line of intersection of planes (1) and (2) is

$x + y + z - 6 + k\,(2x + 3y + 4z + 5) = 0$ ..... (3)

If plane (3) passes through point P, then

$1 + 1 + 1 - 6 + k\,(2 + 3 + 4 + 5) = 0$ or, $k = \dfrac{3}{14}$

From (3) required plane is $20x + 23y + 26z - 69 = 0$

Example # 51 Let planes are $2x + y + 2z = 9$ and $3x - 4y + 12z + 13 = 0$. Which of these bisector planes bisects the acute angle between the given planes. Does origin lie in the acute angle or obtuse angle between the given planes ?

Solution : Given planes are $-2x - y - 2z + 9 = 0$ ..... (1)

and $3x - 4y + 12z + 13 = 0$ ..... (2)

Equations of bisecting planes are $\dfrac{-2x - y - 2z + 9}{\sqrt{(-2)^2 + (-1)^2 + (-2)^2}} = \pm\dfrac{3x - 4y + 12z + 13}{\sqrt{3^2 + (-4)^2 + (12)^2}}$

or, $13\,[-2x - y - 2z + 9] = \pm\, 3\,(3x - 4y + 12z + 13)$

or, $35x + y + 62z = 78$, ... (3) [Taking +ve sign]

and $17x + 25y - 10z = 156$ ..... (4) [Taking – ve sign]

Now $a_1a_2 + b_1b_2 + c_1c_2 = (-2)\,(3) + (-1)\,(-4) + (-2)\,(12)$

$= -6 + 4 - 24 = -26 < 0$

$\therefore$ Bisector of acute angle is given by $35x + y + 62z = 78$

$\because$ $a_1a_2 + b_1b_2 + c_1c_2 < 0$, origin lies in the acute angle between the planes.

Example # 52 : If the planes x – cy – bz = 0, cx – y + az = 0 and bx + ay – z = 0 pass through a straight line, then find the value of $a^2 + b^2 + c^2 + 2abc$.

Solution : Given planes are x – cy – bz = 0 ..... (1)

cx – y + az = 0 ..... (2)

bx + ay – z = 0 ..... (3)

Equation of any plane passing through the line of intersection of planes (1) and (2) may be taken as $x - cy - bz + \lambda(cx - y + az) = 0$

or, $x(1 + \lambda c) - y(c + \lambda) + z(-b + a\lambda) = 0$ ..... (4)

If planes (3) and (4) are the same, then equations (3) and (4) will be identical.

$$\therefore \quad \underset{(i)}{\frac{1+c\lambda}{b}} = \underset{(ii)}{\frac{-(c+\lambda)}{a}} = \underset{(iii)}{\frac{-b+a\lambda}{-1}}$$

From (i) and (ii), $a + ac\lambda = -bc - b\lambda$

or, $\lambda = -\dfrac{(a+bc)}{(ac+b)}$ ..... (5)

From (ii) and (iii),

$c + \lambda = -ab + a^2\lambda$ or $\lambda = \dfrac{-(ab+c)}{1-a^2}$ ..... (6)

From (5) and (6), we have $\dfrac{-(a+bc)}{ac+b} = \dfrac{-(ab+c)}{(1-a^2)}$.

or, $a - a^3 + bc - a^2bc = a^2bc + ac^2 + ab^2 + bc$

or, $a^2bc + ac^2 + ab^2 + a^3 + a^2bc - a = 0$ or, $a^2 + b^2 + c^2 + 2abc = 1$.

Self practice problems:

(59) Find the equation of plane passing through the line of intersection of the planes 2x – 7y + 4z = 3 and 3x – 5y + 4z = 11 and the point (–2, 1, 3).

(60) Find the equations of the planes bisecting the angles between the planes x + 2y + 2z – 3 = 0, 3x + 4y + 12z + 1 = 0 and sepecify the plane which bisects the acute angle between them.

(61) Show that the origin lies in the acute angle between the planes x + 2y + 2z – 9 = 0 and 4x – 3y + 12z + 13 = 0

(62) Prove that the planes 12x – 15y + 16z – 28 = 0, 6x + 6y – 7z – 8 = 0 and 2x + 35y – 39z + 12 = 0 have a common line of intersection.

Answers : (59) 15x – 47y + 28z = 7

(60) 2x + 7y – 5z = 21, 11x + 19y + 31z = 18; 2x + 7y – 5z = 21

Angle between a plane and a line:

(i) If θ is the angle between line $\dfrac{x-x_1}{\ell} = \dfrac{y-y_1}{m} = \dfrac{z-z_1}{n}$ and the plane ax + by + cz + d = 0, then

$$\sin\theta = \left[\frac{a\ell + bm + cn}{\sqrt{(a^2+b^2+c^2)}\sqrt{\ell^2+m^2+n^2}}\right].$$

(ii) Vector form: If θ is the angle between a line $\vec{r} = (\vec{a} + \lambda\vec{b})$ and $\vec{r}\,.\,\vec{n} = d$ then $\sin\theta = \left[\dfrac{\vec{b}\,.\,\vec{n}}{|\vec{b}|\,|\vec{n}|}\right]$.

(iii) Condition for perpendicularity $\quad \dfrac{\ell}{a} = \dfrac{m}{b} = \dfrac{n}{c} \quad \vec{b} \times \vec{n} = 0$

(iv) Condition for parallel $\quad a\ell + bm + cn = 0 \quad \vec{b}\,.\,\vec{n} = 0$

Condition for a line to lie in a plane

(i) **Cartesian form:** Line $\dfrac{x - x_1}{\ell} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n}$ would lie in a plane

$ax + by + cz + d = 0$, if $ax_1 + by_1 + cz_1 + d = 0$ & $a\ell + bm + cn = 0$.

(ii) **Vector form:** Line $\vec{r} = \vec{a} + \lambda\,\vec{b}$ would lie in the plane $\vec{r}\,.\,\vec{n} = d$ if $\vec{b}\,.\,\vec{n} = 0$ & $\vec{a}\,.\,\vec{n} = d$

Example # 53 : Find the distance of the point (1, 0, – 3) from the plane x – y – z = 9 measured parallel to the line $\dfrac{x-2}{2} = \dfrac{y+2}{3} = \dfrac{z-6}{-6}$.

Solution : Given plane is $x - y - z = 9$ ..... (1)

Given line AB is $\dfrac{x-2}{2} = \dfrac{y+2}{3} = \dfrac{z-6}{-6}$ ..... (2)

Equation of a line passing through the point Q(1, 0, – 3) and parallel to line (2) is

$\dfrac{x-1}{2} = \dfrac{y}{3} = \dfrac{z+3}{-6} = r$. ..... (3)

Co-ordinates of any point on line (3) may be taken as

P (2r + 1, 3r, – 6r – 3)

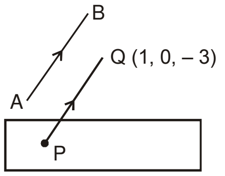
If P is the point of intersection of line (3) and plane (1), then P lies on plane (1),

$\therefore$ (2r + 1) – (3r) – (– 6r – 3) = 9

r = 1

or, $P \equiv (3, 3, -9)$

Distance between points Q (1, 0, – 3) and P (3, 3, – 9)



$PQ = \sqrt{(3-1)^2 + (3-0)^2 + (-9-(-\ 3))^2} = \sqrt{4+9+36} = 7$.

Example # 54 : Find the equation of the plane passing through (1, 2, 0) which contains the line $\dfrac{x+3}{3} = \dfrac{y-1}{4} = \dfrac{z-2}{-2}$.

Solution : Equation of any plane passing through (1, 2, 0) may be taken as

$a(x - 1) + b(y - 2) + c(z - 0) = 0$ ..... (1)

where a, b, c are the direction ratios of the normal to the plane. Given line is

$\dfrac{x+3}{3} = \dfrac{y-1}{4} = \dfrac{z-2}{-2}$ ..... (2)

If plane (1) contains the given line, then

$3a + 4b - 2c = 0$ ..... (3)

Also point (– 3, 1, 2) on line (2) lies in plane (1)

$\therefore$ $a(-3-1) + b(1-2) + c(2-0) = 0$

or, $-4a - b + 2c = 0$ ..... (4)

Solving equations (3) and (4), we get $\frac{a}{8-2} = \frac{b}{8-6} = \frac{c}{-3+16}$

or, $\frac{a}{6} = \frac{b}{2} = \frac{c}{13}$ = k (say). ..... (5)

Substituting the values of a, b and c in equation (1), we get

$6(x-1) + 2(y-2) + 13(z-0) = 0$.

or, $6x + 2y + 13z - 10 = 0$. This is the required equation.

Example # 55 : Find the equation of the projection of the line $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$ on the plane x + 2y + z = 9.

Solution : Let the given line AB be $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$ ..... (1)

Given plane is $x + 2y + z = 9$ ..... (2)

Let DC be the projection of AB on plane (2)

Clearly plane ABCD is perpendicular to plane (2).

Equation of any plane through AB may be taken as (this plane passes through the point (1, – 1, 3) on line AB)

$a(x-1) + b(y+1) + c(z-3) = 0$ ..... (3)

where $2a - b + 4c = 0$ ..... (4)

[$\because$ normal to plane (3) is perpendicular to line (1)]

Since plane (3) is perpendicular to plane (2),

$\therefore$ $a + 2b + c = 0$ ..... (5)

Solving equations (4) & (5), we get $\frac{a}{-9} = \frac{b}{2} = \frac{c}{5}$.

Substituting these values of a, b and c in equation (3), we get

$9(x-1) - 2(y+1) - 5(z-3) = 0$

or, $9x - 2y - 5z + 4 = 0$ ...... (6)

Since projection DC of AB on plane (2) is the line of intersection of plane ABCD and plane (2), therefore equation of DC will be

$$\text{and } \left.\begin{aligned} 9x-2y-5z+4=0 \quad & .....(i) \\ x+2y+z-9=0 \quad & .....(ii) \end{aligned}\right\} \quad ..... (7)$$

Let ℓ, m, n be the direction ratios of the line of intersection of planes (i) and (ii)

$\therefore$ $9\ell - 2m - 5n = 0$ ..... (8) and $\ell + 2m + n = 0$ ..... (9)

$\therefore$ $\frac{\ell}{-2+10} = \frac{m}{-5-9} = \frac{n}{18+2}$ $\Rightarrow$ $\frac{\ell}{4} = \frac{m}{-7} = \frac{n}{10}$

Let any point on line (7) is $(\alpha, \beta, 0)$ $\Rightarrow$ $9\alpha - 2\beta + 4 = 0$

$\alpha + 2\beta - 9 = 0$ $\Rightarrow$ $\alpha = \frac{1}{2}$, $\beta = \frac{17}{4}$

So equation of line is $\frac{x-\frac{1}{2}}{4} = \frac{y-\frac{17}{4}}{-7} = \frac{z-0}{10}$

Self practice problems :

(63) Find the values of a and b for which the line $\frac{x-2}{a} = \frac{y+3}{4} = \frac{z-6}{-2}$ is perpendicular to the plane $3x - 2y + bz + 10 = 0$.

(64) Find the equation of the plane containing the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{3}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$.

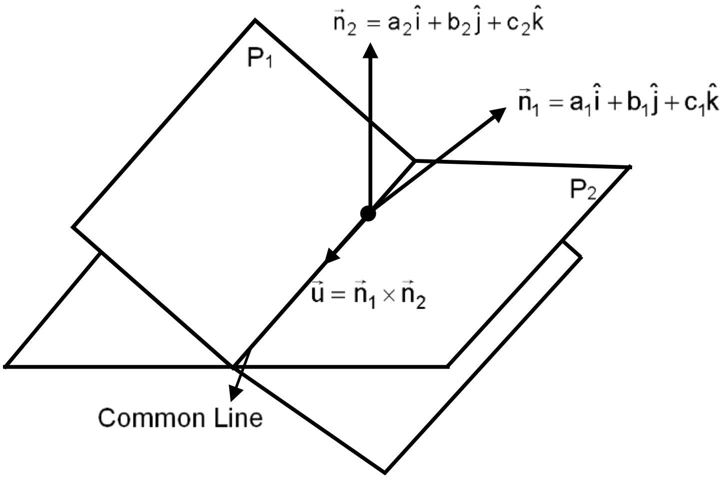
(65) Find the plane containing the line $\frac{x-2}{2} = \frac{y-3}{3} = \frac{z-4}{5}$ and parallel to the line $\frac{x+1}{1} = \frac{y-1}{-2} = \frac{-z+1}{1}$

(66) Show that the line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ & $\frac{x-4}{5} = \frac{y-1}{2} = z$ are intersecting each other. Find their intersection point and the plane containing the line.

Answers : (63) $a = -6, b = 1$ (64) $3x - y - z + 2 = 0$
(65) $13x + 3y - 7z - 7 = 0$ (66) $(-1, -1, -1)$ & $5x - 18y + 11z - 2 = 0$

Non-symmetrical form of line :

A straight line in space is characterised by the intersection of two planes which are not parallel and therefore, the equation of a straight line is a solution of the system constituted by the equations of the two planes, $P_1 \equiv a_1x + b_1y + c_1z + d_1 = 0$ and $P_2 \equiv a_2x + b_2y + c_2z + d_2 = 0$. This form is also known as non-symmetrical form.



To find the equation of the line in symmetrical form, we must know (i) its direction ratios (ii) coordinate of any point on it.

(i) **Direction ratios:** Let ℓ, m, n be the direction ratios of the line. Since the line lies in both the planes, it must be perpendicular to normals of both planes.
So $a_1\ell + b_1m + c_1n = 0$, $a_2\ell + b_2m + c_2n = 0$. From these equations, proportional values of ℓ, m, n can be found by cross-multiplication as $\frac{\ell}{b_1c_2 - b_2c_1} = \frac{m}{c_1a_2 - c_2a_1} = \frac{n}{a_1b_2 - a_2b_1}$

(ii) **Point on the line** – Note that as ℓ, m, n cannot be zero simultaneously, so at least one must be non-zero. Let $a_1b_2 - a_2b_1 \neq 0$, then the line cannot be parallel to xy plane, so it intersect it. Let it intersect xy-plane in $(x_1, y_1, 0)$. Then $a_1x_1 + b_1y_1 + d_1 = 0$ and $a_2x_1 + b_2y_1 + d_2 = 0$.Solving these, we get a point on the line.

Note : If $\ell \neq 0$, then we can take a point on yz-plane as $(0, y_1, z_1)$ and if $m \neq 0$, then we can take a point on xz-plane as $(x_1, 0, z_1)$.

Coplanar lines : Condition of coplanarity if both the lines are in general form Let the lines be

$$ax + by + cz + d = 0 = a'x + b'y + c'z + d' \;\;\&\;\; \alpha x + \beta y + \gamma z + \delta = 0 = \alpha' x + \beta' y + \gamma' z + \delta'$$

They are coplanar if $\begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ \alpha & \beta & \gamma & \delta \\ \alpha' & \beta' & \gamma' & \delta' \end{vmatrix} = 0$

Example # 56 : Find the equation of the line of intersection of planes $4x + 4y - 5z = 12$, $8x + 12y - 13z = 32$ in the symmetric form.

Solution : Given planes are $4x + 4y - 5z - 12 = 0$ ..... (1)

and $2x - 3y + 4z = 5$ ..... (2)

Let ℓ, m, n be the direction ratios of the line of intersection :

then $4\ell - 4m - 3n = 0$ ..... (3)

and $42\ell - 12m + 13n = 0 \therefore \dfrac{\ell}{-8+9} = \dfrac{m}{6-4} = \dfrac{n}{-3+4}$ or, $\dfrac{\ell}{1} = \dfrac{m}{2} = \dfrac{n}{1}$

Hence direction ratios of line of intersection are 1, 2, 1.

Let the line of intersection meet the xy-plane at P $(\alpha, \beta, 0)$.

Then P lies on planes (1) and (2)

$\therefore$ $\alpha - 2\beta = 4$ or, $2\alpha - 3\beta = 5$ ..... (5)

$\alpha = -2$, $\beta = -3$

Hence equation of line of intersection in symmetrical form is $\dfrac{x+2}{1} = \dfrac{y+3}{2} = \dfrac{z}{1}$.

Example # 57 : Find the angle between the lines $x - 3y - 4 = 0$, $4y - z + 5 = 0$ and $x + 3y - 11 = 0$, $2y - z + 6 = 0$.

Solution : Given lines are $\left.\begin{matrix} x - 3y - 4 = 0 \\ 4y - z + 5 = 0 \end{matrix}\right\}$ ..... (1)

and $\left.\begin{matrix} x + 3y - 11 = 0 \\ 2y - z + 6 = 0 \end{matrix}\right\}$ ..... (2)

Let ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 be the direction cosines of lines (1) and (2) respectively

$\because$ line (1) is perpendicular to the normals of each of the planes

$x - 3y - 4 = 0$ and $4y - z + 5 = 0$

$\therefore$ $\ell_1 - 3m_1 + 0.n_1 = 0$ ..... (3) and $0\ell_1 + 4m_1 - n_1 = 0$ ..... (4)

Solving equations (3) and (4), we get $\frac{\ell_1}{3-0} = \frac{m_1}{0-(-1)} = \frac{n_1}{4-0}$

or, $\frac{\ell_1}{3} = \frac{m_1}{1} = \frac{n_1}{4}$ = k (let).

Since line (2) is perpendicular to the normals of each of the planes
x + 3y – 11 = 0 and 2y – z + 6 = 0,

$\therefore$ $\ell_2 + 3m_2 = 0$ ..... (5) and $2m_2 - n_2 = 0$ ..... (6)

$\therefore$ $\ell_2 = -3m_2$ or, $\frac{\ell_2}{-3} = m_2$ and $n_2 = 2m_2$ or, $\frac{n_2}{2} = m_2$.

$\therefore$ $\frac{\ell_2}{-3} = \frac{m_2}{1} = \frac{n_2}{2}$ = t (let).

If θ be the angle between lines (1) and (2), then $\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$

= (3k) (– 3t) + (k) (t) + (4k) (2t) = – 9kt + kt + 8kt = 0 $\therefore$ $\theta = 90°$.

Example # 58 : Show that the lines $\frac{x-3}{2} = \frac{y+1}{-3} = \frac{z+2}{1}$ and $\frac{x-7}{-3} = \frac{y}{1} = \frac{z+7}{2}$ are coplanar. Also find the equation of the plane containing them.

Solution : Given lines are $\frac{x-3}{2} = \frac{y+1}{-3} = \frac{z+2}{1}$ = r (say)..... (1) and $\frac{x-7}{-3} = \frac{y}{1} = \frac{z+7}{2}$ = R (say) ..... (2)

If possible, let lines (1) and (2) intersect at P.
Any point on line (1) may be taken as (2r + 3, – 3r – 1, r – 2) = P (let).
Any point on line (2) may be taken as (– 3R + 7, R, 2R – 7) = P (let).

$\therefore$ 2r + 3 = – 3R + 7 or, 2r + 3R = 4 ..... (3)

Also – 3r – 1 = R or, – 3r – R = 1 ..... (4)

and r – 2 = 2R – 7 or, r – 2R = – 5. ..... (5)

Solving equations (3) and (4), we get, r = – 1, R = 2
Clearly r = – 1, R = 2 satisfies equation (5).
Hence lines (1) and (2) intersect.

$\therefore$ lines (1) and (2) are coplanar.

Equation of the plane containing lines (1) and (2) is $\begin{vmatrix} x-3 & y+1 & z+2 \\ 2 & -3 & 1 \\ -3 & 1 & 2 \end{vmatrix} = 0$

or, (x – 3) (– 6 – 1) – (y + 1) (4 + 3) + (z + 2) (2 – 9) = 0

or, – 7 (x – 3) – 7 (y + 1) – 7 (z + 2) = 0 or, x – 3 + y + 1 + z + 2 = 0 or, x + y + z = 0.

Self practice problems:

(67) Find the equation of the line of intersection of the plane
x – y + 2z = 5, 3x + y + z = 6.

(68) Prove that the three planes 2x + y – 4z – 17 = 0, 3x + 2y – 2z – 25 = 0, 2x – 4y + 3z + 25 = 0 intersect at a point and find its co-ordinates.

Answers : (67) $\frac{4x-11}{-33} = \frac{4y+9}{5} = \frac{z}{1}$ (68) (3, 7, –1)

Exercise-1

✍ **Marked questions are recommended for Revision.**

✍ चिन्हित प्रश्न दोहराने योग्य प्रश्न है।

PART - I : SUBJECTIVE QUESTIONS

भाग - I : विषयात्मक प्रश्न (SUBJECTIVE QUESTIONS)

Section (A) : Position vector, Direction Ratios & Direction cosines

खण्ड (A) : मूलभूत, विभाजन सूत्र और सरल रेखा का समीकरण

A-1. (i) Let position vectors of points A,B and C are $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively. Point D divides line segment BC
internally in the ratio 2 : 1. Find vector $\overrightarrow{AD}$.

(ii) Let ABCD is parallelogram. Position vector of points A,C and D are $\vec{a}$, $\vec{c}$ and $\vec{d}$ respectively. If E divides
line segement AB internally in the ratio 3 : 2 then find vector $\overrightarrow{DE}$.

(iii) Let ABCD is trapezium such that $\overrightarrow{AB} = 3\overrightarrow{DC}$. E divides line segement AB internally in the ratio 2 : 1 and F is mid point of DC. If position vector of A,B and C are $\vec{a}, \vec{b}$ and $\vec{c}$ respectively then find vector $\overrightarrow{FE}$.

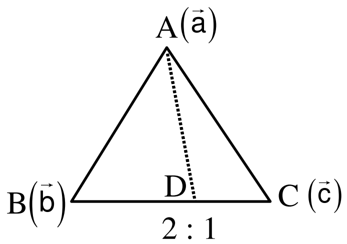
(i) माना कि बिन्दु A,B और C के स्थित सदिश क्रमशः $\vec{a}$, $\vec{b}$ और $\vec{c}$ है। बिन्दु D रेखाखण्ड BC को बिन्दु D पर 2 : 1 में
अन्तः विभाजित करता है। तब सदिश $\overrightarrow{AD}$ ज्ञात कीजिए।

(ii) माना ABCD एक समान्तर चतुर्भज है बिन्दु A,C और D के लिए स्थित सदिश क्रमशः $\vec{a}$, $\vec{c}$ और $\vec{d}$ है। यदि E रेखाखण्ड
AB को 3 : 2 में आन्तरिक विभाजित करता है। तब सदिश $\overrightarrow{DE}$ ज्ञात कीजिए।

(iii) माना ABCD एक समलम्ब चतुर्भज इस प्रकार है कि $\overrightarrow{AB} = 3\overrightarrow{DC}$ है। E रेखाखण्ड AB को 2 : 1 में अन्तःविभाजित करता है।
तथा F, DC का मध्य बिन्दु है। यदि A,B और C के स्थिति सदिश क्रमशः $\vec{a}, \vec{b}$ और $\vec{c}$ है, तो सदिश $\overrightarrow{FE}$ है –

Ans. (i) $\dfrac{2\vec{c}+\vec{b}-3\vec{a}}{3}$ (ii) $\dfrac{3\vec{c}+5\vec{a}-8\vec{d}}{5}$ (iii) $\dfrac{5\vec{b}+\vec{a}-6\vec{c}}{6}$

Sol. **(i)**

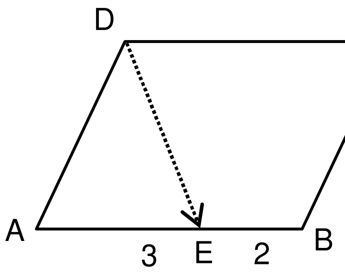


Position vector of D is $\dfrac{2\vec{c}+\vec{b}}{3}$

D का स्थिति सदिश $\dfrac{2\vec{c}+\vec{b}}{3}$ है

vector सदिश $\overrightarrow{AD} = \dfrac{2\vec{c}+\vec{b}}{3} - \vec{a}$

(ii)



Vector सदिश $\overrightarrow{DC} = \vec{c} - \vec{d}$

Vector सदिश $\vec{c} - \vec{d} = \vec{c} - \vec{d}$

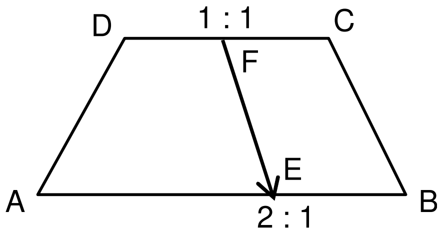
Vector सदिश $\overrightarrow{AE} = \frac{3}{5}\left(\vec{c} - \vec{d}\right)$

Position vector of E is $\vec{a} + \frac{3}{5}(\vec{c} - \vec{d})$

E का स्थिति सदिश $\vec{a} + \frac{3}{5}(\vec{c} - \vec{d})$ है।

vector सदिश $\overrightarrow{DE} = \frac{5\vec{a} + 3\vec{c} - 3\vec{d}}{5} - \vec{d} = \frac{5\vec{a} + 3\vec{c} - 8\vec{d}}{5}$

(iii)



Vector सदिश $\overrightarrow{AB} = \vec{b} - \vec{a}$

Vectorसदिश $\overrightarrow{DF} = \overrightarrow{FC} = \frac{\vec{b} - \vec{a}}{6}$

$\overrightarrow{AF} = \overrightarrow{AC} + \overrightarrow{CF} = (\vec{c} - \vec{a}) + \frac{1}{6}\left(\vec{a} - \vec{b}\right)$

vector सदिश $\overrightarrow{AE} = \frac{2}{3}\left(\vec{b} - \vec{a}\right)$

$\overrightarrow{FE} = \overrightarrow{AE} - \overrightarrow{AF} = \frac{2}{3}\left(\vec{b} - \vec{a}\right) - \left((\vec{c} - \vec{a}) + \frac{1}{6}\left(\vec{a} - \vec{b}\right)\right)$

$= \frac{5}{6}\left(\vec{b} - \vec{a}\right) + \vec{a} - \vec{c}$

$= \frac{5\vec{b} + \vec{a} - 6\vec{c}}{6}$

A-2. In a ΔABC में , $\overrightarrow{AB} = 6\hat{i}+3\hat{j}+3\hat{k}$; $\overrightarrow{AC} = 3\hat{i} - 3\hat{j} + 6\hat{k}$ **[16JM120001]**

D and D′ are points trisections of side BC

Find $\overrightarrow{AD}$ and $\overrightarrow{AD'}$.

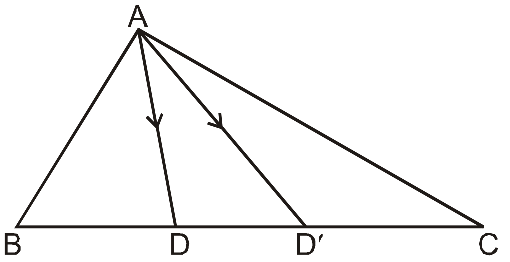
ΔABC में , $\overrightarrow{AB} = 6\hat{i}+3\hat{j}+3\hat{k}$; $\overrightarrow{AC} = 3\hat{i} - 3\hat{j} + 6\hat{k}$

D और D′ भुजा BC के त्रिविभाजन के बिन्दु है।

$\overrightarrow{AD}$ तथा $\overrightarrow{AD'}$ ज्ञात कीजिए।

Ans. $5\hat{i} + \hat{j} + 4\hat{k}$, $4\hat{i} - \hat{j} + 5\hat{k}$.

Sol.



$$\overrightarrow{AD} = \frac{2\cdot\overrightarrow{AB} + 1\cdot\overrightarrow{AC}}{3} = \frac{12\hat{i} + 6\hat{j} + 6\hat{k} + 3\hat{i} - 3\hat{j} + 6\hat{k}}{3}$$

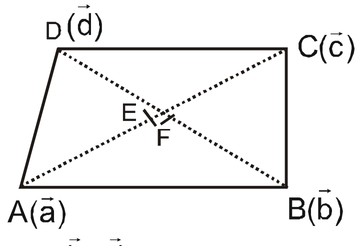
$$= \frac{15\hat{i} + 3\hat{j} + 12\hat{k}}{3} = 5\hat{i} + \hat{j} + 4\hat{k}$$

$$\overrightarrow{AD'} = \frac{1\cdot\overrightarrow{AB} + 2\cdot\overrightarrow{AC}}{3} = \frac{6\hat{i} + 3\hat{j} + 3\hat{k} + 6\hat{i} - 6\hat{j} + 12\hat{k}}{3} = 4\hat{i} - \hat{j} + 5\hat{k}.$$

A-3. If ABCD is a quadrilateral, E and F are the mid-points of AC and BD respectively, then prove that $\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{CD} = 4\overrightarrow{EF}$

यदि ABCD एक चतुर्भुज है तथा E और F क्रमशः भुजा AC तथा BD के मध्य बिन्दु है, तो प्रदर्शित कीजिए कि $\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{CD} = 4\overrightarrow{EF}$

Sol. $E \rightarrow \dfrac{\vec{a}+\vec{c}}{2}$



$F \rightarrow \dfrac{\vec{b}+\vec{d}}{2}$

$\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{CD}$

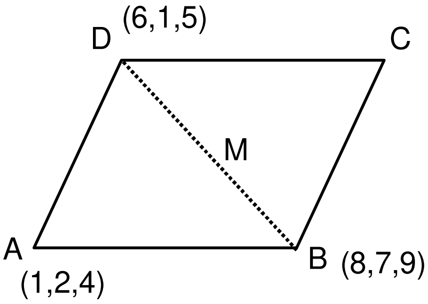
$$= \vec{b} - \vec{a} + \vec{d} - \vec{a} + \vec{b} - \vec{c} + \vec{d} - \vec{c} = 2[\vec{b} + \vec{d} - \vec{a} - \vec{c}] = 4\left[\frac{\vec{b}+\vec{d}}{2} - \frac{(\vec{a}+\vec{c})}{2}\right] = 4\overrightarrow{EF}$$

A-4. Let ABCD is parallelogram where A = (1,2,4) , B = (8,7,9) and D = (6,1,5) . Find direction cosines of line AC

माना ABCD एक समान्तर चतुर्भज है जहाँ A = (1,2,4) , B = (8,7,9) और D = (6,1,5) तब रेखा AC की दिक् कोज्याऐं ज्ञात कीजिए।

Ans. $\left(\frac{6}{7},\frac{2}{7},\frac{3}{7}\right)$, $\left(\frac{-6}{7},\frac{-2}{7},\frac{-3}{7}\right)$

Sol. Mid point of BD is M (7,4,7)



Direction ratio of AM is (6,2,3)

direction cosines of AC is $\left(\frac{6}{7},\frac{2}{7},\frac{3}{7}\right)$ or $\left(\frac{-6}{7},\frac{-2}{7},\frac{-3}{7}\right)$

A-5. Let one vertex of cuboid whose faces are parallel to coordinate planes is (1, 2, 3). If mid-point of one of the edge in (0, 0,0) then find possible

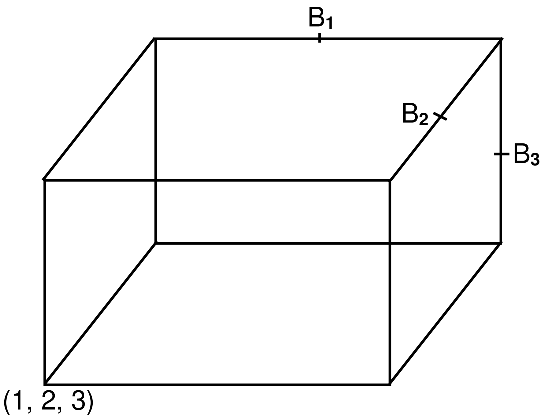
(i) total surface area of cuboid. (ii) length of body diagonal.

माना कि घनाभ जिसके फलक निर्देशांक समतलों के समान्तर है, का एक शीर्ष (1, 2, 3) है यदि किनारे के एक का मध्य बिन्दु (0, 0, 0) है। तब संभावित है।

(i) घनाभ का कुल पृष्ठीय क्षेत्रफल (ii) विकर्ण की लम्बाई

Ans. (i) 40, 38, 32 (ii) $\sqrt{41}$, $\sqrt{26}$, $\sqrt{17}$

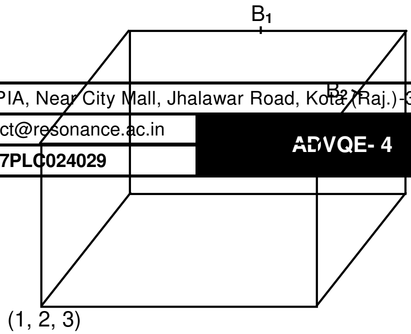
Sol.



(0, 0, 0) can be either B_1 or B_2 or B_3

Length of sides of cuboid are either 1, 2, 6 or 1,4,3 or 2, 2, 3

$\Rightarrow$ Length of body diagonal of cuboid can be $\sqrt{41}$, $\sqrt{26}$, $\sqrt{17}$



$\Rightarrow$ Surface area of cuboid can be 40, 38, 32

Hindi.

(0, 0, 0) B_1 या B_2 या B_3 हो सकता है।

घनाभ की भुजा की लम्बाई 1, 2, 6 या 1,4,3 या 2, 2, 3 है।

$\Rightarrow$ धनाभ के विकर्ण की लम्बाई $\sqrt{41}$, $\sqrt{26}$, $\sqrt{17}$

$\Rightarrow$ घनाभ का पृष्ठीय क्षेत्रफल 40, 38, 32 है।

A-6. Find the direction cosines ℓ, m, n of line which are connected by the relations $\ell + m + n = 0$, $2mn + 2m\ell - n\ell = 0$.

रेखा की दिक्कोज्याऐं ℓ, m, n ज्ञात कीजिए जबकि ये $\ell + m + n = 0$ और $2mn + 2m\ell - n\ell = 0$ द्वारा सम्बन्धित है।

Ans. $-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$ or या $\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}$

Sol. $n = -(\ell + m)$ ....(1) $\quad 2mn + 2ml - nl = 0$ ....(2)

$-2k(\ell + m) + 2m\ell + (\ell + m)\ell = 0$

$\Rightarrow \quad -2m\ell - 2m^2 + 2m\ell + \ell^2 + m\ell = 0 \quad \Rightarrow \quad \ell^2 + m\ell - 2m^2 = 0 \quad \Rightarrow \quad \frac{\ell}{m} = 1, \ -2$

Case - I :

We have यहाँ $\frac{\ell}{m} = 1 \Rightarrow \ell = m$ So इसलिए $n = -2\ell$ So अतः, $\ell : m : n = 1 : 1 : -2$.

$\Rightarrow$ direction cosines are दिक्कोज्याएँ $\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}$ or या $-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$ है।

Case - II :

We have यहाँ $\frac{\ell}{m} = -2 \quad \Rightarrow \quad \ell = -2m \quad \Rightarrow \quad \ell : m : n = -2 : 1 : 1$

henceअतः direction cosines are दिक्कोज्याएँ $\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}$ or या $\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}$ होगी।

Section (B) : Dot Product, Projection and Cross Product
खण्ड (B) : अदिश गुणन, प्रक्षेप और सदिश गुणन

B-1. Show that the points A, B, C with position vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ respectively are the vertices of a right angled triangle. Also find the remaining angles of the triangle.

दर्शाइये कि बिन्दु A, B, C है जिनके स्थिति सदिश $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ और $3\hat{i} - 4\hat{j} - 4\hat{k}$ समकोण त्रिभुज के शीर्ष है तथा त्रिभुज के शेष कोण भी ज्ञात कीजिए –

Ans. $\angle A = \cos^{-1}\sqrt{\frac{35}{41}}$, $\angle B = \cos^{-1}\sqrt{\frac{6}{41}}$, $\angle C = \frac{\pi}{2}$

Solution : We have,

$\overline{AB} = (\hat{i} - 3\hat{j} - 5\hat{k}) - (2\hat{i} - \hat{j} + \hat{k}) = -\hat{i} - 2\hat{j} - 6\hat{k}$

$\vec{BC} = (3\hat{i}-4\hat{j}-4\hat{k}) - (\hat{i}-3\hat{j}-5\hat{k}) = 2\hat{i}-\hat{j}+\hat{k}$

and, $\vec{CA} = (2\hat{i}-\hat{j}+\hat{k}) - (3\hat{i}-4\hat{j}-4\hat{k}) = -\hat{i}+3\hat{j}+5\hat{k}$

Since $\vec{AB}+\vec{BC}+\vec{CA} = (-\hat{i}-2\hat{j}-6\hat{k}) + (2\hat{i}-\hat{j}+\hat{k}) + (-\hat{i}+3\hat{j}+5\hat{k}) = \vec{0}$

So A, B and C are the vertices of a triangle.

Now, $\vec{BC}\,.\,\vec{CA} = (2\hat{i}-\hat{j}+\hat{k})\,.\,(-\hat{i}+3\hat{j}+5\hat{k}) = -2-3+5 = 0$

$\Rightarrow \vec{BC} \perp \vec{CA} \Rightarrow \angle BCA = \frac{\pi}{2} \qquad \Rightarrow$ ABC is a right angled triangle.

Since A is the angle between the vectors $\vec{AB}$ and $\vec{AC}$. Therefore

$$\cos A = \frac{\vec{AB}\,.\,\vec{AC}}{|\vec{AB}||\vec{AC}|} = \frac{(-\hat{i}-2\hat{j}-6\hat{k})\,.\,(\hat{i}-3\hat{j}-5\hat{k})}{\sqrt{(-1)^2+(-2)^2+(-6)^2}\,\sqrt{1^2+(-3)^2+(-5)^2}}$$

$$= \frac{-1+6+30}{\sqrt{1+4+36}\sqrt{1+9+25}} = \frac{35}{\sqrt{41}\sqrt{35}} = \sqrt{\frac{35}{41}} \Rightarrow A = \cos^{-1}\sqrt{\frac{35}{41}}$$

$$\cos B = \frac{\vec{BA}\,.\,\vec{BC}}{|\vec{BA}||\vec{BC}|} = \frac{(\hat{i}+2\hat{j}+6\hat{k})\,.\,(2\hat{i}-\hat{j}+\hat{k})}{\sqrt{1^2+2^2+6^2}\,\sqrt{2^2+(-1)^2+(1)^2}}$$

$$\Rightarrow \cos B = \frac{2-2+6}{\sqrt{41}\sqrt{6}} = \sqrt{\frac{6}{41}} \Rightarrow B = \cos^{-1}\sqrt{\frac{6}{41}}$$

हल : यहाँ, $\vec{AB} = (\hat{i}-3\hat{j}-5\hat{k}) - (2\hat{i}-\hat{j}+\hat{k}) = -\hat{i}-2\hat{j}-6\hat{k}$

$\vec{BC} = (3\hat{i}-4\hat{j}-4\hat{k}) - (\hat{i}-3\hat{j}-5\hat{k}) = 2\hat{i}-\hat{j}+\hat{k}$

तथा, $\vec{CA} = (2\hat{i}-\hat{j}+\hat{k}) - (3\hat{i}-4\hat{j}-4\hat{k}) = -\hat{i}+3\hat{j}+5\hat{k}$

चूंकि $\vec{AB}+\vec{BC}+\vec{CA} = (-\hat{i}-2\hat{j}-6\hat{k}) + (2\hat{i}-\hat{j}+\hat{k}) + (-\hat{i}+3\hat{j}+5\hat{k}) = \vec{0}$

इसलिए A, B तथा C त्रिभुज के शीर्ष है।

अब, $\vec{BC}\,.\,\vec{CA} = (2\hat{i}-\hat{j}+\hat{k})\,.\,(-\hat{i}+3\hat{j}+5\hat{k}) = -2-3+5 = 0$

$\Rightarrow \vec{BC} \perp \vec{CA} \Rightarrow \angle BCA = \frac{\pi}{2} \qquad \Rightarrow$ ABC समकोण त्रिभुज है।

$\vec{AB}$ और $\vec{AC}$ के मध्य कोण A है इसलिए

$$\cos A = \frac{\vec{AB}\,.\,\vec{AC}}{|\vec{AB}||\vec{AC}|} = \frac{(-\hat{i}-2\hat{j}-6\hat{k})\,.\,(\hat{i}-3\hat{j}-5\hat{k})}{\sqrt{(-1)^2+(-2)^2+(-6)^2}\,\sqrt{1^2+(-3)^2+(-5)^2}}$$

$$= \frac{-1+6+30}{\sqrt{1+4+36}\sqrt{1+9+25}} = \frac{35}{\sqrt{41}\sqrt{35}} = \sqrt{\frac{35}{41}} \Rightarrow A = \cos^{-1}\sqrt{\frac{35}{41}}$$

$$\cos B = \frac{\vec{BA}\,.\,\vec{BC}}{|\vec{BA}||\vec{BC}|} = \frac{(\hat{i}+2\hat{j}+6\hat{k})\,.\,(2\hat{i}-\hat{j}+\hat{k})}{\sqrt{1^2+2^2+6^2}\,\sqrt{2^2+(-1)^2+(1)^2}}$$

$$\Rightarrow \cos B = \frac{2-2+6}{\sqrt{41}\sqrt{6}} = \sqrt{\frac{6}{41}} \Rightarrow B = \cos^{-1}\sqrt{\frac{6}{41}}$$

B-2. If $\vec{a}, \vec{b}, \vec{c}$ are three mutually perpendicular vectors of equal magnitude, prove that $\vec{a}+\vec{b}+\vec{c}$ is equally inclined with vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

यदि $\vec{a}, \vec{b}, \vec{c}$ तीन परस्पर लम्बवत समान परिमाण के है। तब सिद्ध कीजिए कि $\vec{a}+\vec{b}+\vec{c}$, सदिश $\vec{a}, \vec{b}$ तथा $\vec{c}$ से बराबर कोण पर झुकी हुई है।

Solution : Let $|\vec{a}| = |\vec{b}| = |\vec{c}| = \lambda$ (say). Since $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors, therefore $\vec{a}\ .\ \vec{b} = \vec{b}\ .\ \vec{c} = \vec{c}\ .\ \vec{a} = 0$ ....(i)

Now, $\left|\vec{a}+\vec{b}+\vec{c}\right|^2 = \vec{a}\ .\ \vec{a} + \vec{b}\ .\ \vec{b} + \vec{c}\ .\ \vec{c} + 2\vec{a}\ .\ \vec{b} + 2\vec{b}\ .\ \vec{c} + 2\vec{c}\ .\ \vec{a}$

$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 = 3\lambda^2\ [\because |\vec{a}| = |\vec{b}| = |\vec{c}| = \lambda]$

$\therefore \quad |\vec{a}+\vec{b}+\vec{c}| = \sqrt{3}\lambda$ ....(ii)

Suppose $\vec{a}+\vec{b}+\vec{c}$ makes angles $\theta_1, \theta_2, \theta_3$ with $\vec{a}, \vec{b}$ and $\vec{c}$ respectively. Then,

$$\cos\theta_1 = \frac{\vec{a}\ .\ (\vec{a}+\vec{b}+\vec{c})}{|\vec{a}||\vec{a}+\vec{b}+\vec{c}|} = \frac{\vec{a}\ .\ \vec{a}+\vec{a}\ .\ \vec{b}+\vec{a}\ .\ \vec{c}}{|\vec{a}||\vec{a}+\vec{b}+\vec{c}|} = \frac{|\vec{a}|^2}{|\vec{a}||\vec{a}+\vec{b}+\vec{c}|}$$

$$= \frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|} = \frac{\lambda}{\sqrt{3}\lambda} = \frac{1}{\sqrt{3}} \text{ [Using (ii)]} \Rightarrow \theta_1 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$\therefore$ Similarly, $\theta_2 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ and $\theta_3 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Because $\theta_1 = \theta_2 = \theta_3$, so $\vec{a}+\vec{b}+\vec{c}$ is equally inclineded with $\vec{a}, \vec{b}$ and $\vec{c}$

हल : माना $|\vec{a}| = |\vec{b}| = |\vec{c}| = \lambda$ (say). चूंकि $\vec{a}, \vec{b}, \vec{c}$ लम्बवत सदिश है

$\vec{a}\ .\ \vec{b} = \vec{b}\ .\ \vec{c} = \vec{c}\ .\ \vec{a} = 0$ ....(i)

अब, $\left|\vec{a}+\vec{b}+\vec{c}\right|^2 = \vec{a}\ .\ \vec{a} + \vec{b}\ .\ \vec{b} + \vec{c}\ .\ \vec{c} + 2\vec{a}\ .\ \vec{b} + 2\vec{b}\ .\ \vec{c} + 2\vec{c}\ .\ \vec{a}$

$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 = 3\lambda^2\ [\because |\vec{a}| = |\vec{b}| = |\vec{c}| = \lambda]$

$\therefore \quad |\vec{a}+\vec{b}+\vec{c}| = \sqrt{3}\lambda$ ....(ii)

मानाकि $\vec{a}+\vec{b}+\vec{c}$, $\vec{a}, \vec{b}, \vec{c}$ के साथ $\theta_1, \theta_2, \theta_3$ कोण बनाते है,

$$\cos\theta_1 = \frac{\vec{a}\ .\ (\vec{a}+\vec{b}+\vec{c})}{|\vec{a}||\vec{a}+\vec{b}+\vec{c}|} = \frac{\vec{a}\ .\ \vec{a}+\vec{a}\ .\ \vec{b}+\vec{a}\ .\ \vec{c}}{|\vec{a}||\vec{a}+\vec{b}+\vec{c}|} = \frac{|\vec{a}|^2}{|\vec{a}||\vec{a}+\vec{b}+\vec{c}|}$$

$$= \frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|} = \frac{\lambda}{\sqrt{3}\lambda} = \frac{1}{\sqrt{3}} \text{ [(ii) से] } \Rightarrow \theta_1 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$\therefore$ इसीप्रकार, $\theta_2 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ और $\theta_3 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

परन्तु $\theta_1 = \theta_2 = \theta_3$, इसलिए $\vec{a}+\vec{b}+\vec{c}$, $\vec{a}, \vec{b}$ और $\vec{c}$ बराबर झुकी हुई है।

B-3. (i) Find the projection of $\vec{b}+\vec{c}$ on $\vec{a}$ where $\vec{a} = \hat{i}+2\hat{j}+\hat{k}$, $\vec{b} = \hat{i}+3\hat{j}+\hat{k}$ and $\vec{c} = \hat{i}+\hat{k}$.

$\vec{b}+\vec{c}$ का $\vec{a}$ पर प्रक्षेप ज्ञात कीजिए जहाँ $\vec{a} = \hat{i}+2\hat{j}+\hat{k}$, $\vec{b} = \hat{i}+3\hat{j}+\hat{k}$ तथा $\vec{c} = \hat{i}+\hat{k}$ है।

(ii) Find the projection of the line segment joining (2, – 1, 3) and (4, 2, 5) on a line which makes equal acute angles with co-ordinate axes.
बिन्दु (2, – 1, 3) और (4, 2, 5) को मिलाने वाली सरल रेखा का, उस सरल रेखा पर प्रक्षेप ज्ञात करो जो अक्षों के साथ समान न्यूनकोण बनाती है।

(iii) P and Q are the points (–1, 2, 1) and (4, 3, 5) respectively. Find the projection of PQ on a line which makes angles of 120º and 135º with y and z axes respectively and an acute angle with x-axis.
बिन्दु P तथा Q क्रमशः (–1, 2, 1) तथा (4, 3, 5) हैं। PQ का उस सरल रेखा पर प्रक्षेप ज्ञात कीजिए जो y तथा z अक्ष से क्रमशः 120º तथा 135º का कोण बनाती है तथा x-अक्ष से न्यूनकोण बनाती है। **[16JM120008]**

Ans. **(i)** $\frac{10}{\sqrt{6}}$ (ii) $\frac{7}{\sqrt{3}}$ (iii) $2\sqrt{2} - 2$

Sol. (i) $\vec{b}+\vec{c} = \hat{i}+3\hat{j}+\hat{k}+\hat{i}+\hat{k} = 2\hat{i}+3\hat{j}+2\hat{k}$

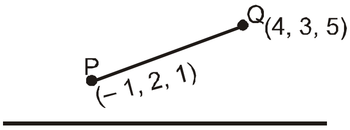
Projection of $\vec{b}+\vec{c}$ on $\vec{a}$ = $\dfrac{(\vec{b}+\vec{c})\cdot\vec{a}}{|\vec{a}|} = \dfrac{(2\hat{i}+3\hat{j}+2\hat{k})\cdot(\hat{i}+2\hat{j}+\hat{k})}{\sqrt{(1)^2+(2)^2+(1)^2}} = \dfrac{2+6+2}{\sqrt{1+4+1}} = \dfrac{10}{\sqrt{6}}$

Hindi (i) $\vec{b}+\vec{c} = \hat{i}+3\hat{j}+\hat{k}+\hat{i}+\hat{k} = 2\hat{i}+3\hat{j}+2\hat{k}$

$\vec{b}+\vec{c}$ का $\vec{a}$ पर प्रक्षेप = $\dfrac{(\vec{b}+\vec{c})\cdot\vec{a}}{|\vec{a}|} = \dfrac{(2\hat{i}+3\hat{j}+2\hat{k})\cdot(\hat{i}+2\hat{j}+\hat{k})}{\sqrt{(1)^2+(2)^2+(1)^2}} = \dfrac{2+6+2}{\sqrt{1+4+1}} = \dfrac{10}{\sqrt{6}}$

(iii) $\because$ $m = \cos 120° = \dfrac{-1}{2}$; $n = \cos 135° = \dfrac{-1}{\sqrt{2}}$ $\because$ $\ell^2 + m^2 + n^2 = 1$

$\Rightarrow$ $\ell^2 + \dfrac{1}{4} + \dfrac{1}{2} = 1$ $\Rightarrow$ $\ell^2 = \dfrac{1}{4}$ $\Rightarrow$ $\ell = \dfrac{1}{2}$ as acute angle with x-axis



$\ell = \dfrac{1}{2},\ m = -\dfrac{1}{2},\ n = -\dfrac{1}{\sqrt{2}}$

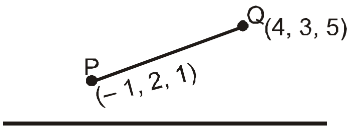
$\therefore$ required projection = $|(x_2 - x_1)\,\ell + (y_2 - y_1)\,m + (z_2 - z_1)n|$

$= |5 \times 1/2 + 1 \times (-1/2) + 4 \times (-1/\sqrt{2})| = \left|\dfrac{5}{2} - \dfrac{1}{2} - 2\sqrt{2}\right| = 2(\sqrt{2}-1)$

Hindi $\because$ $m = \cos 120° = \dfrac{-1}{2}$; $n = \cos 135° = \dfrac{-1}{\sqrt{2}}$ $\because$ $\ell^2 + m^2 + n^2 = 1$

$\Rightarrow$ $\ell^2 + \dfrac{1}{4} + \dfrac{1}{2} = 1$ $\Rightarrow$ $\ell^2 = \dfrac{1}{4}$

$\Rightarrow$ $\ell = \dfrac{1}{2}$ जैसा कि रेखा x-अक्ष के साथ न्यून कोण बनाती है।



$\ell = \dfrac{1}{2},\ m = -\dfrac{1}{2},\ n = -\dfrac{1}{\sqrt{2}}$

$\therefore$ अभीष्ट प्रक्षेप = $|(x_2 - x_1)\,\ell + (y_2 - y_1)\,m + (z_2 - z_1)n|$

$= |5 \times 1/2 + 1 \times (-1/2) + 4 \times (-1/\sqrt{2})| = \left|\dfrac{5}{2} - \dfrac{1}{2} - 2\sqrt{2}\right| = 2(\sqrt{2}-1)$

B-4. Prove that $\left(\dfrac{\vec{a}}{\vec{a}^2} - \dfrac{\vec{b}}{\vec{b}^2}\right)^2 = \left(\dfrac{\vec{a}-\vec{b}}{|\vec{a}|\ |\vec{b}|}\right)^2$

सिद्ध कीजिए : $\left(\dfrac{\vec{a}}{\vec{a}^2} - \dfrac{\vec{b}}{\vec{b}^2}\right)^2 = \left(\dfrac{\vec{a}-\vec{b}}{|\vec{a}|\ |\vec{b}|}\right)^2$

Sol.. LHS $\dfrac{|\vec{a}|^2}{|\vec{a}|^4} + \dfrac{|\vec{b}|^2}{|\vec{b}|^4} - \dfrac{2\vec{a}\cdot\vec{b}}{|\vec{a}|^2\ |\vec{b}|^2} \Rightarrow \dfrac{1}{|\vec{a}|^2} + \dfrac{1}{|\vec{b}|^2} - \dfrac{2\vec{a}\cdot\vec{b}}{|\vec{a}|^2|\vec{b}|^2}$

RHS $\dfrac{|\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}\cdot\vec{b}}{|\vec{a}|^2\,|\vec{b}|^2} \Rightarrow \dfrac{1}{|\vec{a}|^2} + \dfrac{1}{|\vec{b}|^2} - \dfrac{2\vec{a}\cdot\vec{b}}{|\vec{a}|^2|\vec{b}|^2}$ .Hence Proved.

Hindi बायाँ पक्ष $\dfrac{|\vec{a}|^2}{|\vec{a}|^4} + \dfrac{|\vec{b}|^2}{|\vec{b}|^4} - \dfrac{2\vec{a}\cdot\vec{b}}{|\vec{a}|^2\ |\vec{b}|^2} \Rightarrow \dfrac{1}{|\vec{a}|^2} + \dfrac{1}{|\vec{b}|^2} - \dfrac{2\vec{a}\cdot\vec{b}}{|\vec{a}|^2|\vec{b}|^2}$

दायाँ पक्ष $\dfrac{|a|^2+|b|^2-2\vec{a}\ .\ \vec{b}}{|a|^2\,|b|^2} \Rightarrow \dfrac{1}{|a|^2}+\dfrac{1}{|b|^2}-\dfrac{2\vec{a}\ .\ \vec{b}}{|a|^2|b|^2}$. अतः सिद्ध हुआ।

B-5. If $\vec{a}$, $\vec{b}$ are two unit vectors and θ is the angle between them, then show that:
यदि $\vec{a}$, $\vec{b}$ दो इकाई सदिश है और θ उनके मध्य कोण है, तो प्रदर्शित कीजिए कि

(a) $\sin\dfrac{\theta}{2}=\dfrac{1}{2}\left|\vec{a}-\vec{b}\right|$ (b) $\cos\dfrac{\theta}{2}=\dfrac{1}{2}\left|\vec{a}+\vec{b}\right|$

Sol. (a) $|\vec{a}-\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2-2(\vec{a}\ .\ \vec{b}) \Rightarrow |\vec{a}-\vec{b}|^2=1+1-2(1)\times(1)\times\cos\theta$

$\Rightarrow |\vec{a}-\vec{b}|^2=1+1-2(1)\times(1)\times\cos\theta \Rightarrow |\vec{a}-\vec{b}|^2=2\ [1-\cos\theta] \Rightarrow |\vec{a}-\vec{b}|^2=4\sin^2\dfrac{\theta}{2}$

$\Rightarrow \sin\dfrac{\theta}{2}=\dfrac{1}{2}|\vec{a}-\vec{b}|$

(b) $|\vec{a}+\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2+2(\vec{a}\ .\ \vec{b}) \Rightarrow |\vec{a}+\vec{b}|^2=1+1+2\times1\times1\cos\theta$

$\Rightarrow |\vec{a}+\vec{b}|^2=2[1+\cos\theta]=4\cos^2\dfrac{\theta}{2} \Rightarrow \cos\dfrac{\theta}{2}=\dfrac{1}{2}|\vec{a}+\vec{b}|$

B-6. If two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}|=2$, $|\vec{b}|=1$ and $\vec{a}\cdot\vec{b}=1$, then find the value of $(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b})$. **[16JM120004]**
यदि दो सदिश $\vec{a}$ तथा $\vec{b}$ इस प्रकार है कि $|\vec{a}|=2$, $|\vec{b}|=1$ तथा $\vec{a}\cdot\vec{b}=1$ तो $(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b})$ का मान ज्ञात कीजिए।

Ans. 0

Sol. Here, two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}|=2$, $|\vec{b}|=1$ and $\vec{a}\cdot\vec{b}=1$,
Now, $(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b}) = 6|\vec{a}|^2-10\,\vec{a}\cdot\vec{b}+21\,\vec{a}\cdot\vec{b}-35|\vec{b}|^2$
$= 6(2)^2-10\times1+(21\times1)-35(1)^2$
$= 24-10+21-35 = 45-45 = 0$

Hindi. यहाँ, दो सदिश $\vec{a}$ तथा $\vec{b}$ इस प्रकार है कि $|\vec{a}|=2$, $|\vec{b}|=1$ तथा $\vec{a}\cdot\vec{b}=1$,
अब, $(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b}) = 6|\vec{a}|^2-10\,\vec{a}\cdot\vec{b}+21\,\vec{a}\cdot\vec{b}-35|\vec{b}|^2$
$= 6(2)^2-10\times1+(21\times1)-35(1)^2 = 24-10+21-35 = 45-45 = 0$

B-7. For any two vectors $\vec{u}$ & $\vec{v}$, prove that **[16JM120003]**
(a) $(\vec{u}.\vec{v})^2+|\vec{u}\times\vec{v}|^2=|\vec{u}|^2|\vec{v}|^2$ & (b) $(1+|\vec{u}|^2)(1+|\vec{v}|^2)=(1-\vec{u}.\vec{v})^2+|\vec{u}+\vec{v}+(\vec{u}\times\vec{v})|^2$
किन्ही दो सदिशों $\vec{u}$ और $\vec{v}$ के लिए सिद्ध करो कि –
(a) $(\vec{u}.\vec{v})^2+|\vec{u}\times\vec{v}|^2=|\vec{u}|^2|\vec{v}|^2$ & (b) $(1+|\vec{u}|^2)(1+|\vec{v}|^2)=(1-\vec{u}.\vec{v})^2+|\vec{u}+\vec{v}+(\vec{u}\times\vec{v})|^2$

Sol. (a) $|\vec{v}|^2\,|\vec{u}|^2\cos^2\theta+|\vec{v}|^2\,|\vec{u}|^2\sin^2\theta=|\vec{v}|^2\,|\vec{u}|^2$
(b) LHS $1+|\vec{v}|^2+|\vec{u}|^2+|\vec{u}|^2\,|\vec{v}|^2$
RHS $1+|u|^2\,|\vec{v}|^2\cos^2\theta-2|\vec{u}|\,|\vec{v}|\cos\theta+|\vec{u}|^2+|\vec{v}|^2+|\vec{u}|^2|\vec{v}|^2\sin^2\theta+2\vec{u}.\vec{v}+0+0$
$=1+|\vec{v}|^2+|\vec{u}|^2+|\vec{u}|^2\,|\vec{v}|^2$
LHS = RHS
Hence Proved. (इति सिद्धम)

B-8. If the three successive vertices of a parallelogram have the position vectors as, A (– 3, – 2, 0); B (3, – 3, 1) and C (5, 0, 2). Then find
(a) position vector of the fourth vertex D

B-9. If $\vec{a},\vec{b},\vec{c}$ are three vectors such that $|\vec{a}| = 5$, $|\vec{b}| = 12$ and $|\vec{c}| = 13$, and $\vec{a}+\vec{b}+\vec{c}=\vec{o}$, find the value of $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$. **[16JM120005]**

यदि $\vec{a},\vec{b}$ तथा $\vec{c}$ तीन ऐसे सदिश हैं कि $|\vec{a}| = 5$, $|\vec{b}| = 12$ तथा $|\vec{c}| = 13$, तथा $\vec{a}+\vec{b}+\vec{c}=\vec{o}$ है, तो $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$ का मान ज्ञात कीजिए।

Ans. -169

Sol. Consider $\vec{a}+\vec{b}+\vec{c} = \vec{o} \Rightarrow \left(\vec{a}+\vec{b}+\vec{c}\right) = \vec{o}^{\,2}$

$\Rightarrow \vec{a}^2+\vec{b}^2+\vec{c}^2 + 2\vec{a}.\vec{b}+2\vec{b}.\vec{c}+2\vec{c}.\vec{a} = 0 \Rightarrow 25 + 144 + 169 + 2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}) = 0$

$\Rightarrow 2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}) = -338 \Rightarrow \vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a} = -169$

B-10. ABCD is a parallelogram in which $\overrightarrow{AB} = 3\hat{i} - 6\hat{j} + 3\hat{k}$ and $\overrightarrow{AD} = 6\hat{i} + 6\hat{j} + 3\hat{k}$. P is a point of AB such that AP : PB = 1 : 2 and Q is a point on BC such that BQ : QC = 2 : 1. Find angle between DQ and PC.

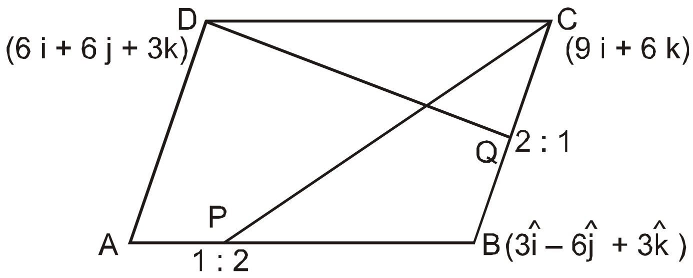
ABCD एक समान्तर चतुर्भुज है जिसमें $\overrightarrow{AB} = 3\hat{i} - 6\hat{j} + 3\hat{k}$ तथा $\overrightarrow{AD} = 6\hat{i} + 6\hat{j} + 3\hat{k}$ है। P, AB पर एक बिन्दु इस प्रकार है कि AP : PB = 1 : 2 तथा BC पर बिन्दु Q इस प्रकार है कि BQ : QC = 2 : 1 तब DQ और PC के मध्य कोण ज्ञात कीजिए।

Ans. $\theta = \cos^{-1}\dfrac{2}{3\sqrt{713}}$

Sol. Let A be the origin $\overrightarrow{AP} = \hat{i} - 2\hat{j} + \hat{k}$

माना A मूल बिन्दु है तब $\overrightarrow{AP} = \hat{i} - 2\hat{j} + \hat{k}$

$$\overrightarrow{AQ} = \frac{18\hat{i} + 12\hat{k} + 3\hat{i} - 6\hat{j} + 3\hat{k}}{3} = 7\hat{i} - 2\hat{j} + 5\hat{k}$$



$\overrightarrow{DQ} = 7\hat{i} - 2\hat{j} + 5\hat{k} - 6\hat{i} - 6\hat{j} - 3\hat{k} = \hat{i} - 8\hat{j} + 2\hat{k}$

$\overrightarrow{PC} = 9\hat{i} + 6\hat{k} - (\hat{i} - 2\hat{j} + \hat{k}) \Rightarrow \overrightarrow{DQ} \, . \, \overrightarrow{PC} = |\overrightarrow{DQ}|\,|\overrightarrow{PC}|\cos\theta$

$8 - 16 + 15 = \sqrt{69}.\sqrt{93}\cos\theta \Rightarrow 2 = 3\sqrt{23}.\sqrt{31}\cos\theta \Rightarrow \theta = \cos^{-1}\dfrac{2}{3\sqrt{713}}$

B-11. Prove using vectors : If two medians of a triangle are equal, then it is isosceles.

सदिश विधि से दर्शाइये कि यदि त्रिभुज की दो माध्यिकाएं बराबर है तब यह समद्विबाहु है।

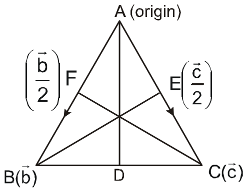
Sol. Let ABC be a triangle and let BE and CF be two equal medians. Taking A as the origin, let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively. Then,

P.V. of E = $\dfrac{1}{2}\vec{c}$ and P.V. of F = $\dfrac{1}{2}\vec{b}$

$\therefore \quad \overrightarrow{BE} = \dfrac{1}{2}(\vec{c} - 2\vec{b}) \Rightarrow \overrightarrow{CF} = \dfrac{1}{2}(\vec{b} - 2\vec{c})$

Now, BE = CF $\Rightarrow |\overrightarrow{BE}| = |\overrightarrow{CF}|$

$\Rightarrow |\overrightarrow{BE}|^2 = |\overrightarrow{CF}|^2 \Rightarrow \left|\dfrac{1}{2}(\vec{c} - 2\vec{b})\right|^2 = \left|\dfrac{1}{2}(\vec{b} - 2\vec{c})\right|^2$



$\Rightarrow \quad \frac{1}{4}|\vec c - 2\vec b|^2 = \frac{1}{4}|\vec b - 2\vec c|^2 \Rightarrow |\vec c - 2\vec b|^2 = |\vec b - 2\vec c|^2$

$\Rightarrow \quad (\vec c - 2\vec b) \cdot (\vec c - 2\vec b) = (\vec b - 2\vec c) \cdot (\vec b - 2\vec c)$

$\Rightarrow \quad \vec c \cdot \vec c - 4\vec b \cdot \vec c + 4\vec b \cdot \vec b = \vec b \cdot \vec b - 4\vec b \cdot \vec c + 4\vec c \cdot \vec c$

$\Rightarrow \quad |\vec c|^2 - 4\vec b \cdot \vec c + 4|\vec b|^2 = |\vec b|^2 - 4\vec b \cdot \vec c + 4|\vec c|^2 \Rightarrow 3|\vec b|^2 = 3|\vec c|^2 \Rightarrow |\vec b|^2 = |\vec c|^2$

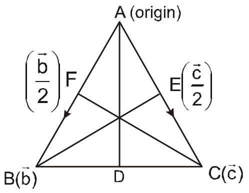
$\Rightarrow$ AB = AC Hence triangle ABC is an isosceles triangle.

Hindi. माना ABC एक त्रिभुज है तथा BE तथा CF दो बराबर माध्यिकाएं है। A को मूलबिन्दू लेते हुए माना B और C के स्थिति सदिश क्रमशः $\vec b$ और $\vec c$ है तब E का स्थिति सदिश = $\frac{1}{2}\vec c$ तथा F का स्थिति सदिश = $\frac{1}{2}\vec b$

$\therefore \quad \overrightarrow{BE} = \frac{1}{2}(\vec c - 2\vec b) \quad \Rightarrow \quad \overrightarrow{CF} = \frac{1}{2}(\vec b - 2\vec c)$

अब, BE = CF $\Rightarrow |\overrightarrow{BE}| = |\overrightarrow{CF}|$

$\Rightarrow \quad |\overrightarrow{BE}|^2 = |\overrightarrow{CF}|^2 \Rightarrow \left|\frac{1}{2}(\vec c - 2\vec b)\right|^2 = \left|\frac{1}{2}(\vec b - 2\vec c)\right|^2$



$\Rightarrow \quad \frac{1}{4}|\vec c - 2\vec b|^2 = \frac{1}{4}|\vec b - 2\vec c|^2 \Rightarrow |\vec c - 2\vec b|^2 = |\vec b - 2\vec c|^2$

$\Rightarrow \quad (\vec c - 2\vec b) \cdot (\vec c - 2\vec b) = (\vec b - 2\vec c) \cdot (\vec b - 2\vec c)$

$\Rightarrow \quad \vec c \cdot \vec c - 4\vec b \cdot \vec c + 4\vec b \cdot \vec b = \vec b \cdot \vec b - 4\vec b \cdot \vec c + 4\vec c \cdot \vec c$

$\Rightarrow \quad |\vec c|^2 - 4\vec b \cdot \vec c + 4|\vec b|^2 = |\vec b|^2 - 4\vec b \cdot \vec c + 4|\vec c|^2 \Rightarrow 3|\vec b|^2 = 3|\vec c|^2 \Rightarrow |\vec b|^2 = |\vec c|^2$

$\Rightarrow$ AB = AC अतः ABC समद्विबाहु त्रिभुज है।

B-12. (i) Find the angle between the lines whose direction cosines are given by the equations :
$3\ell + m + 5n = 0$ and $6\,mn - 2n\ell + 5\,\ell m = 0$

(ii) Find the angle between the lines whose direction cosines are given by $\ell + m + n = 0$ and $\ell^2 + m^2 = n^2$.

(i) उन सरल रेखाओं के मध्य कोण ज्ञात करो जिनकी दिक् कोज्याये निम्न समीकरणों को संतुष्ट कर रही है–
$3\ell + m + 5n = 0$ और $6\,mn - 2n\ell + 5\,\ell m = 0$

(ii) उन रेखाओं के मध्य कोण ज्ञात कीजिए जिनकी दिक्कोज्याएँ निम्न सम्बन्धों से दी जाती है,
$\ell + m + n = 0$ तथा $\ell^2 + m^2 = n^2$.

Ans. (i) $\cos^{-1}\left(\frac{1}{6}\right)$ (ii) 60º

Sol. (i) We have :
$3l + m + 5n = 0$ ...(1)
$6mn - 2nl + 5lm = 0$ ...(2)

From (1), $m = -(3l + 5n)$ ...(3)
Putting the value in (2), we get
$-6(3\ell + 5n)n - 2n\ell - 15\ell^2 - 25n\ell = 0 \Rightarrow -18\ell n - 30n^2 - 2n\ell - 15\ell^2 - 25n\ell = 0$

$-30n^2 - 45n\ell - 15\ell^2 = 0 \Rightarrow 2n^2 + 3\ell n + \ell^2 = 0$ [Divide by (–15)]

$(2n + \ell)(n + \ell) = 0$

Either $2n + \ell = 0$ or $n + \ell = 0$

(I) when $2n + \ell = 0 \Rightarrow \ell = -2n$
from (3), $m = -(-6n + 5n) = n$
(II) when $n + \ell = 0 \Rightarrow \ell = -n$
from (3), $m = -(-3n + 5n) = -2n$
$\therefore$ Direction ratios of two lines are
$-2n, n, n$ and $-n, -2n, n$
$-2, 1, 1$ and $1, 2, -1$

Angle between two lines $\Rightarrow \cos\theta = \left|\dfrac{(-2)(1)+(1)(2)+(1)(-1)}{\sqrt{(-2)^2+1^2+1^2}\ \sqrt{1^2+2^2+(-1)^2}}\right|$.

Hence $\theta = \cos^{-1}\left(\dfrac{1}{6}\right)$

Hindi. दिया है
$3l + m + 5n = 0$ ...(1)
$6mn - 2nl + 5lm = 0$ ...(2)
समीकरण (1) से, $m = -(3l + 5n)$ ...(3)
निम्न का मान समीकरण (2) में रखने पर
$-6(3\ell + 5n)n - 2n\ell - 15\ell^2 - 25n\ell = 0 \Rightarrow -18\ell n - 30n^2 - 2n\ell - 15\ell^2 - 25n\ell = 0$

$-30n^2 - 45n\ell - 15\ell^2 = 0 \Rightarrow 2n^2 + 3\ell n + \ell^2 = 0$ [(–15) से भाग देने पर]

$(2n + \ell)(n + \ell) = 0$

या तो जब $2n + \ell = 0$ या $n + \ell = 0$

(I) जब $2n + \ell = 0 \Rightarrow \ell = -2n$

समीकरण (3), से $m = -(-6n + 5n) = n$

(II) जब $n + \ell = 0 \Rightarrow \ell = -n$

समीकरण (3) से, $m = -(-3n + 5n) = -2n$

अतः दोनो सरल रेखाओं के दिक् अनुपात निम्न होगे
$-2n, n, n$ तथा $-n, -2n, n$
$-2, 1, 1$ तथा $1, 2, -1$

दोनो सरल रेखाओं के मध्य कोण $\cos\theta = \left|\dfrac{(-2)(1)+(1)(2)+(1)(-1)}{\sqrt{(-2)^2+1^2+1^2}\ \sqrt{1^2+2^2+(-1)^2}}\right|$

$\Rightarrow$ अतः $\theta = \cos^{-1}\left(\dfrac{1}{6}\right)$

(ii) $\ell + m + n = 0$ ....(1) $\ell^2 + m^2 = n^2$ ....(2)

Put $n = -(\ell + m)$ in (2) $\Rightarrow \ell^2 + m^2 = \ell^2 + m^2 + 2\ell m \Rightarrow \ell m = 0$

Case-I if $\ell = 0$; $m \neq 0$ then from (1) $m = -n$

$\therefore \dfrac{\ell}{0} = \dfrac{m}{1} = \dfrac{n}{-1}$ direction cosine are : $0, \dfrac{1}{\sqrt{2}}, \dfrac{-1}{\sqrt{2}}$

Case-II if $\ell \neq 0$; m = 0, then from (1), $\ell = -n$

$\therefore \quad \frac{\ell}{1} = \frac{m}{0} = \frac{n}{-1} \quad \therefore \quad$ direction cosine are : $\frac{1}{\sqrt{2}}, \ 0, \ \frac{-1}{\sqrt{2}}$

Let θ be the angle between the lines

$\cos\theta = 0 + 0 + \frac{1}{2} \quad \Rightarrow \quad \cos\theta = \frac{1}{2} \quad\quad \theta = \frac{\pi}{3}$

Hindi $\ell + m + n = 0$ ....(1) $\quad \ell^2 + m^2 = n^2$ ....(2)

(2) में $n = -(\ell + m)$ रखने पर $\Rightarrow \quad \ell^2 + m^2 = \ell^2 + m^2 + 2\ell m \quad \ell m = 0$

Case-I यदि $\ell = 0$; $m \neq 0$ तो (1) से $m = -n \quad \therefore \quad \frac{\ell}{0} = \frac{m}{1} = \frac{n}{-1}$

दिक् कोज्याऐं है $0, \ \frac{1}{\sqrt{2}}, \ \frac{-1}{\sqrt{2}}$

Case-II यदि $\ell \neq 0$; m = 0, तो (1) से $\ell = -n \ \therefore \ \frac{\ell}{1} = \frac{m}{0} = \frac{n}{-1}$ दिक् कोज्याऐं है : $\frac{1}{\sqrt{2}}, \ 0, \ \frac{-1}{\sqrt{2}}$

माना θ रेखाओं के मध्य कोण है $\quad \cos\theta = 0 + 0 + \frac{1}{2}$

$\cos\theta = \frac{1}{2} \quad\quad \theta = \frac{\pi}{3}$

B-13. Position vectors of A, B, C are given by $\vec{a}, \ \vec{b}, \ \vec{c}$ where $\vec{a}\times\vec{b}+\vec{b}\times \ \vec{c}+\vec{c}\times\vec{a} = 0$. If $\vec{AC} = 2\hat{i}-3\hat{j}+6\hat{k}$ then find $\vec{BC}$ if BC = 14. **[DRN2037]**

A, B, C के स्थिति सदिश क्रमशः $\vec{a}, \ \vec{b}, \ \vec{c}$ है जबकि $\vec{a}\times\vec{b}+\vec{b}\times \ \vec{c}+\vec{c}\times\vec{a} = 0$. यदि $\vec{AC} = 2\hat{i}-3\hat{j}+6\hat{k}$ तब $\vec{BC}$ ज्ञात कीजिये यदि BC = 14.

Ans. $\pm(4\hat{i}-6\hat{j}+12\hat{k})$

Sol. $\vec{a}\times\vec{b}-\vec{c}\times\vec{b}+ \ \vec{c}\times\vec{a}-\vec{c}\times\vec{c} \quad \Rightarrow \quad (\vec{a}-\vec{c})\times\vec{b}+\vec{c}\times(\vec{a}-\vec{c}) = 0 \quad \Rightarrow \quad (\vec{a}-\vec{c})\times(\vec{b}-\vec{c}) = 0$

$\vec{CA}\times\vec{CB} = 0 \quad \therefore \quad \vec{BC}$ is | | to $\vec{AC} \quad \Rightarrow \quad \vec{BC} = \pm 14\left(\frac{2\hat{i}-3\hat{j}+6\hat{k}}{7}\right) = \pm(4\hat{i}-6\hat{j}+12\hat{k})$

B-14. A vector $\vec{c}$ is perpendicular to the vectors $2\hat{i}+3\hat{j}-\hat{k}, \hat{i}-2\hat{j}+3\hat{k}$ and satisfies the condition $\vec{c}.\left(2\hat{i} - \hat{j} + \hat{k}\right) + 6 = 0$. Find the vector $\vec{c}$

एक सदिश $\vec{c}$ सदिशों $2\hat{i}+3\hat{j}-\hat{k}$ एवं $\hat{i}-2\hat{j}+3\hat{k}$ के लम्बवत् है और $\vec{c}.\left(2\hat{i} - \hat{j} + \hat{k}\right) + 6 = 0$ को संतुष्ट करता है। सदिश $\vec{c}$ को ज्ञात कीजिए।

Ans. $3\left(-\hat{i}+\hat{j}+\hat{k}\right)$

Sol. Let $\vec{c} = \alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}$

$\vec{c} \perp 2\hat{i}+3\hat{j}-\hat{k} \quad \Rightarrow \quad 2\alpha + 3\beta - \gamma = 0$ ....(1)

$\vec{c} \perp \hat{i}-2\hat{j}+3\hat{k} \quad \Rightarrow \quad \alpha - 2\beta + 3\gamma = 0$ ....(2)

and given condition

$\vec{c}.(2\hat{i}-\hat{j}+\hat{k})+6=0 \quad \Rightarrow \quad 2\alpha-\beta+\gamma+6=0$ ....(3)

Solving (1), (2) & (3)

$\alpha = -3 \quad\quad \beta = 3 \quad\quad \gamma = 3$. Hence $\quad \vec{c} = 3(-\hat{i}+\hat{j}+\hat{k})$

Hindi. मानाकि $\vec c = \alpha\hat i + \beta\hat j + \gamma\hat k$

$\vec c \perp 2\hat i + 3\hat j - \hat k \;\Rightarrow\; 2\alpha + 3\beta - \gamma = 0$ ....(1)

$\vec c \perp \hat i - 2\hat j + 3\hat k \;\Rightarrow\; \alpha - 2\beta + 3\gamma = 0$ ....(2)

और दिया गया प्रतिबन्ध

$\vec c.(2\hat i - \hat j + \hat k) + 6 = 0 \;\Rightarrow\; 2\alpha - \beta + \gamma + 6 = 0$ ....(3)

(1), (2) और (3) को हल करने पर $\alpha = -3$ $\beta = 3$

$\gamma = 3$. अत: $\vec c = 3(-\hat i + \hat j + \hat k)$

B-15. (a) Show that the perpendicular distance of the point $\vec c$ from the line joining $\vec a$ and $\vec b$ is

$$\frac{\left|\vec b\times\vec c+\vec c\times\vec a+\vec a\times\vec b\right|}{\left|\vec b-\vec a\right|}.$$ **[16JM120007]**

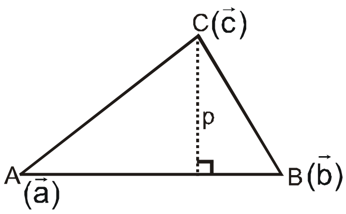
(b) Given a parallelogram ABCD with area 12 sq. units. A straight line is drawn through the mid point M of the side BC and the vertex A which cuts the diagonal BD at a point 'O'. Use vectors to determine the area of the quadrilateral OMCD.

(a) प्रदर्शित कीजिए कि $\vec a$ एवं $\vec b$ को मिलाने वाली रेखा से बिन्दु $\vec c$ की लम्बवत् दूरी

$$\frac{\left|\vec b\times\vec c+\vec c\times\vec a+\vec a\times\vec b\right|}{\left|\vec b-\vec a\right|}$$ है।

(b) एक समान्तर चतुर्भुज ABCD का क्षेत्रफल 12 वर्ग इकाई दिया गया है। भुजा BC के मध्य बिन्दु M और शीर्ष A से एक सरल रेखा खीचीं गई है जो विकर्ण BD को बिन्दु 'O' पर काटती है। चतुर्भुज OMCD का क्षेत्रफल सदिश विधि की सहायता से ज्ञात कीजिए।

Ans. (b) 5 unit sq. वर्ग इकाई



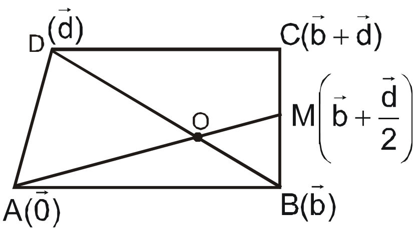
Sol. (a)

Let $\perp$ distance of $\vec c$ from line joining $\vec a$ and $\vec b$ is p. Now $\Delta = \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \frac{1}{2}\left|\overrightarrow{AB}\right|\times p$

$$\Rightarrow \quad p = \frac{|\overrightarrow{AB}\times\overrightarrow{AC}|}{|\overrightarrow{AB}|} = \frac{|(\vec b-\vec a)\times(\vec c-\vec a)|}{|\vec b-\vec a|} = \frac{|\vec a\times\vec b+\vec b\times\vec c+\vec c\times\vec a|}{|\vec b-\vec a|}$$

(b) Equation of line AM is

$$\vec r = \lambda\left(\vec b + \frac{\vec d}{2}\right)$$



Equation of line BD is

$$\vec r = \vec b + \mu\ (\vec d - \vec b)$$

to obtain point of intersection

$$\lambda\left(\vec{b}+\frac{\vec{d}}{2}\right) = \vec{b}+\mu(\vec{d}-\vec{b}) \Rightarrow \quad \lambda = 1-\mu \ \& \ \frac{\lambda}{2}=\mu \quad \Rightarrow \quad \lambda = 1-\frac{\lambda}{2} \text{ or } \lambda = \frac{2}{3}$$

hence point O is $\frac{2}{3}\left(\vec{b}+\frac{\vec{d}}{2}\right)$. Area OMCD = Area OMC + Area OCD

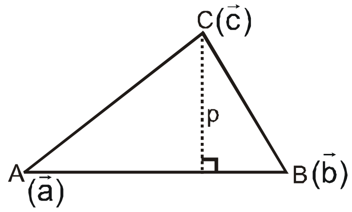
$$= \frac{1}{2}\left|\frac{1}{3}\left(\vec{b}+\frac{\vec{d}}{2}\right)\times\left(\frac{\vec{b}}{3}+\frac{2\vec{d}}{3}\right)\right| + \frac{1}{2}\left|\left(\frac{\vec{b}}{3}+\frac{2\vec{d}}{3}\right)\times\left(\frac{-2}{3}\vec{b}+\frac{2}{3}\vec{d}\right)\right|$$

$$= \frac{1}{2}\left|\frac{1}{9}\left(\vec{b}\times 2\vec{d}+\frac{\vec{d}}{2}\times\vec{b}\right)\right| + \frac{1}{2}\left|\frac{1}{9}(\vec{b}\times 2\vec{d}-4\vec{d}\times\vec{b})\right|$$

$$= \frac{1}{18}\left|\frac{3}{2}\ \vec{b}\times\vec{d}\right| + \frac{1}{18}|6\vec{b}\times\vec{d}| = \frac{1}{18}\times\frac{15}{2}|\vec{b}\times\vec{d}|$$

$$= \frac{15}{18\times 2}\times 12 = 5 \text{ sq. units}$$

Hindi (a)



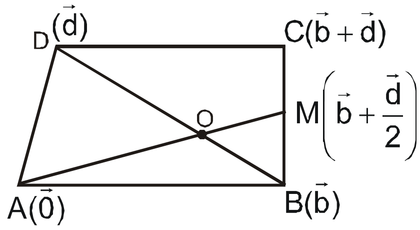
मानाकि $\vec{c}$ की $\vec{a}$ और $\vec{b}$ को जोड़ने वाली रेखा से लम्बवत दूरी p है।

अब $\Delta = \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \frac{1}{2}\left|\overrightarrow{AB}\right|\times p$

$$\Rightarrow \quad p = \frac{|\overrightarrow{AB}\times\overrightarrow{AC}|}{|\overrightarrow{AB}|} = \frac{|(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})|}{|\vec{b}-\vec{a}|} = \frac{|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|}{|\vec{b}-\vec{a}|}$$

(b) रेखा AM का समीकरण है

$$\vec{r} = \lambda\left(\vec{b}+\frac{\vec{d}}{2}\right)$$



रेखा BD का समीकरण है $\vec{r} = \vec{b}+\mu\ (\vec{d}-\vec{b})$

प्रतिच्छेद बिन्दु प्राप्त करने के लिए

$$\lambda\left(\vec{b}+\frac{\vec{d}}{2}\right) = \vec{b}+\mu(\vec{d}-\vec{b}) \Rightarrow \quad \lambda = 1-\mu \ \& \ \frac{\lambda}{2}=\mu \quad \Rightarrow \quad \lambda = 1-\frac{\lambda}{2} \text{ or } \lambda = \frac{2}{3}$$

अतः बिन्दु O, $\frac{2}{3}\left(\vec{b}+\frac{\vec{d}}{2}\right)$ है।

OMCD का क्षैत्रफल = OMC का क्षैत्रफल + OCD का क्षैत्रफल

$$= \frac{1}{2}\left|\frac{1}{3}\left(\vec{b}+\frac{\vec{d}}{2}\right)\times\left(\frac{\vec{b}}{3}+\frac{2\vec{d}}{3}\right)\right| + \frac{1}{2}\left|\left(\frac{\vec{b}}{3}+\frac{2\vec{d}}{3}\right)\times\left(\frac{-2}{3}\vec{b}+\frac{2}{3}\vec{d}\right)\right|$$

$$= \frac{1}{2}\left|\frac{1}{9}\left(\vec{b}\times 2\vec{d}+\frac{\vec{d}}{2}\times\vec{b}\right)\right| + \frac{1}{2}\left|\frac{1}{9}(\vec{b}\times 2\vec{d}-4\vec{d}\times\vec{b})\right|$$

$$= \frac{1}{18}\left|\frac{3}{2}\ \vec{b}\times\vec{d}\right| + \frac{1}{18}|6\vec{b}\times\vec{d}| = \frac{1}{18}\times\frac{15}{2}|\vec{b}\times\vec{d}| = \frac{15}{18\times 2}\times 12 = 5$$ वर्ग इकाई

B-16. P, Q are the mid-points of the non-parallel sides BC and AD of a trapezium ABCD. Show that $\Delta APD = \Delta CQB$.

P, Q समलम्ब चतुर्भुज ABCD की असमान्तर भुजाओं BC और AD के मध्य बिन्दु है। दर्शाइये कि $\Delta APD = \Delta CQB$.

Solution : Let $\overrightarrow{AB} = \vec{b}$ and $\overrightarrow{AD} = \vec{d}$

Now DC is parallel to AB $\Rightarrow$ there exists a scalar t such that $\overrightarrow{DC} = t\,\overrightarrow{AB} = t\vec{b}$

$\therefore$ $\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \vec{d}+t\,\vec{b}$

The position vectors of P and Q are $\frac{1}{2}(\vec{b}+\vec{d}+t\,\vec{b})$ and $\frac{1}{2}\vec{d}$ respectively.

Now $2\Delta\ \overrightarrow{APD} = \overrightarrow{AP} \times \overrightarrow{AD}$

$= \frac{1}{2}(\vec{b}+\vec{d}+t\,\vec{b}) \times \vec{d} = \frac{1}{2}(1+t)(\vec{b}\times\vec{d})$

Also $2\Delta\overrightarrow{CQB} = \overrightarrow{CQ} \times \overrightarrow{CB} = \left[\frac{1}{2}\vec{d}-(\vec{d}+t\vec{b})\right] \times [\vec{b}-(\vec{d}+t\,\vec{b})]$

$= \left[-\frac{1}{2}\vec{d}-t\,\vec{b}\right]\times\left[-\vec{d}+(1-t)\,\vec{b}\right] = -\frac{1}{2}(1-t)\ (\vec{d}\times\vec{b})+t\ (\vec{b}\times\vec{d})$

$= \frac{1}{2}(1-t+2t)\ (\vec{b}\times\vec{d}) = \frac{1}{2}(1+t)\ (\vec{b}\times\vec{d}) = 2\Delta\overrightarrow{APD}$ Hence Prove.

Solution : माना $\overrightarrow{AB} = \vec{b}$ तथा $\overrightarrow{AD} = \vec{d}$

अब DC , AB के समान्तर है $\Rightarrow$ अदिश t विद्यमान है जबकि $\overrightarrow{DC} = t\,\overrightarrow{AB} = t\vec{b}$

$\therefore$ $\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \vec{d}+t\,\vec{b}$

P और Q के स्थिति सदिश $\frac{1}{2}(\vec{b}+\vec{d}+t\,\vec{b})$ तथा $\frac{1}{2}\vec{d}$ है।

अब $2\Delta\ \overrightarrow{APD} = \overrightarrow{AP} \times \overrightarrow{AD}$

$= \frac{1}{2}(\vec{b}+\vec{d}+t\,\vec{b}) \times \vec{d} = \frac{1}{2}(1+t)(\vec{b}\times\vec{d})$

तथा $2\Delta\overrightarrow{CQB} = \overrightarrow{CQ} \times \overrightarrow{CB} = \left[\frac{1}{2}\vec{d}-(\vec{d}+t\vec{b})\right] \times [\vec{b}-(\vec{d}+t\,\vec{b})]$

$= \left[-\frac{1}{2}\vec{d}-t\,\vec{b}\right]\times\left[-\vec{d}+(1-t)\,\vec{b}\right] = -\frac{1}{2}(1-t)\ (\vec{d}\times\vec{b})+t\ (\vec{b}\times\vec{d})$

$= \frac{1}{2}(1-t+2t)\ (\vec{b}\times\vec{d}) = \frac{1}{2}(1+t)\ (\vec{b}\times\vec{d}) = 2\Delta\overrightarrow{APD}$ अतः सिद्ध हुआ।

Section (C) : Line
खण्ड (C) : रेखा

C-1. Find the coordinates of the point when the line through (3, 2, 5) and (–2, 3, –5) crosses the xy plane.

उस बिन्दु के निर्देशांक ज्ञात कीजिए जबकि रेखा, xy समतल को काटती हुई बिन्दुओं (3, 2, 5) और (–2, 3, –5) से गुजरती है।

Ans $\left(\frac{1}{2},\frac{5}{2},0\right)$

Sol. (x, y, 0) divides line segment joining (3, 2, 5) and (–2, 3, –5) in the ratio 1 : 1 internally.

$\Rightarrow x = \frac{3-2}{2} = \frac{1}{2}$, $y = \frac{2+3}{2} = \frac{5}{2}$

Sol. (x, y, 0) बिन्दुओं (3, 2, 5) और (–2, 3, –5) को मिलाने वाली रेखा को 1 : 1 मे आन्तरिक विभाजित करती है।

$\Rightarrow x = \frac{3-2}{2} = \frac{1}{2}$, $y = \frac{2+3}{2} = \frac{5}{2}$

C-2. Find the foot of the perpendicular from (1, 6, 3) on the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$.

बिन्दु (1, 6, 3) से सरल रेखा $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ पर डाले गये लम्ब पाद के निर्देशांक ज्ञात कीजिए।

Ans. (1, 3, 5)

Sol. Any point on the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ k (say) ...(1)

is P(k, 2k + 1, 3k + 2)

Let it be the foot of $\perp$ from Q(1, 6, 3) on the line (1)

Then d.r.s. of PQ are k – 1, 2k + 1 – 6, 3k + 2 – 3 i.e., k – 1, 2k – 5, 3k – 1

As PQ is $\perp$ to the line (1), so

$(k-1) \times 1 + (2k-5) \times 2 + (3k-1) \times 3 = 0 \Rightarrow k - 1 + 4k - 10 + 9k - 3 = 0 \Rightarrow 14k - 14 = 0 \Rightarrow k = 1$.

$\therefore$ The foot of the $\perp$ is given by P(1, 2 × 1 + 1, 3 × 1 + 2) = (1, 3, 5).

Hindi. सरल रेखा $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ = k (माना) ...(1)

पर स्थित P(k, 2k + 1, 3k + 2) बिन्दु होगा

माना यही बिन्दु Q(1, 6, 3) से सरल रेखा (1) पर डाले गये लम्ब पाद के निर्देशांक ज्ञात करो।

तो PQ के दिक् अनुपात है। k – 1, 2k + 1 – 6, 3k + 2 – 3 अर्थात् k – 1, 2k – 5, 3k – 1

चूँकि PQ सरल रेखा (1) के लम्बवत् है, इसलिये $(k-1) \times 1 + (2k-5) \times 2 + (3k-1) \times 3 = 0$

$\Rightarrow k - 1 + 4k - 10 + 9k - 3 = 0 \Rightarrow 14k - 14 = 0 \Rightarrow k = 1$.

अतः लम्ब पाद का निर्देशंक P(1, 2 × 1 + 1, 3 × 1 + 2) = (1, 3, 5) होगें।

C-3. (i) Find the cartesian form of the equation of a line whose vector form is given by .

$$\vec{r} = 2\hat{i} - \hat{j} + 4\hat{k} + \lambda(\hat{i} + \hat{j} - 2\hat{k})$$

(i) एक रेखा का सदिश समीकरण $\vec{r} = 2\hat{i} - \hat{j} + 4\hat{k} + \lambda(\hat{i} + \hat{j} - 2\hat{k})$ है, तो इसका कार्तीय समीकरण ज्ञात कीजिए।

(ii) Find the vector form of the equation of a line whose cartesian form is given by $\frac{2x-4}{1} = \frac{3y+6}{2} = \frac{-6z+6}{1}$. **[16JM120009]**

(ii) रेखा के समीकरण का सदिश रूप लिखिए जिसका कार्तीय रूप $\frac{2x-4}{1} = \frac{3y+6}{2} = \frac{-6z+6}{1}$ है।

Ans. (i) $\frac{x-2}{1} = \frac{y+1}{1} = \frac{z-4}{-2}$ (ii) $\vec{r} = (2\hat{i} - 2\hat{j} + \hat{k}) + \lambda(3\hat{i} + 4\hat{j} - \hat{k})$

Sol. (i) $\because$ $\vec{r} = 2\hat{i} - \hat{j} + 4\hat{k} + \lambda(\hat{i} + \hat{j} - 2\hat{k})$

Let $(x\hat{i} + y\hat{j} + z\hat{k})$ be a point lying on the line

मानाकि दी गई रेखा पर स्थित कोई बिन्दु $(x\hat{i} + y\hat{j} + z\hat{k})$ है।

$\therefore$ $x\hat{i} + y\hat{j} + z\hat{k} = \hat{i}(2+\lambda) + \hat{j}(\lambda - 1) + (4 - 2\lambda) \Rightarrow$ $x - 2 = \lambda$; $y + 1 = \lambda$;

$\therefore$ equation of line is $\dfrac{x-2}{1} = \dfrac{y+1}{1} = \dfrac{z-4}{-2}$

$\therefore$ रेखा का समीकरण $\dfrac{x-2}{1} = \dfrac{y+1}{1} = \dfrac{z-4}{-2}$ है।

C-4. Find the distance between points of intersection of

Lines $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{4}$ & $\dfrac{x-4}{5} = \dfrac{y-1}{2} = z$ and

Lines $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j})$ & $\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$

निम्नलिखित रेखाओं के प्रतिच्छेद बिन्दुओं के मध्य दूरी ज्ञात कीजिए –

रेखाओं $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{4}$ एवं $\dfrac{x-4}{5} = \dfrac{y-1}{2} = z$ तथा

रेखाओं $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j})$ & $\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$

Ans. $\sqrt{26}$

Sol. $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{4} = r$ (Let) ... (1)

$\Rightarrow$ $(2r + 1,\ 3r + 2,\ 4r + 3)$ represents any point on (1)

$\dfrac{x-4}{5} = \dfrac{y-1}{2} = \dfrac{z-0}{1}$ ... (2)

To find point of intersection of (1) and (2)

$\dfrac{2r+1-4}{5} = \dfrac{3r+2-1}{2} = \dfrac{4r+3}{1} \Rightarrow \dfrac{2r-3}{5} = \dfrac{3r+1}{2} = \dfrac{4r+3}{1} \Rightarrow 4r - 6 = 15r + 5 \Rightarrow 11r = -11 \Rightarrow r = -1$

$\therefore$ point of intersection of (1) and (2) is $(-1, -1, -1)$

$\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j})$ ... (1) $\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$ ... (2)

For their point of intersection

$3\lambda + 1 = 4 + 2\mu \Rightarrow 3\lambda - 2\mu - 3 = 0$ ... (3)

$1 - \lambda = 0 \Rightarrow \lambda = 1$ ... (4)

and $-1 = -1 + 3\mu$ $\mu = 0$

$\therefore$ point of intersection is $(4, 0, -1)$

$\therefore$ required distance $= \sqrt{(4+1)^2 + 1 + 0} = \sqrt{25+1} = \sqrt{26}$

Hindi. $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{4} = r$ (माना) ... (1)

$\Rightarrow$ $(2r + 1,\ 3r + 2,\ 4r + 3)$ रेखा (1) पर स्थित किसी बिन्दु को प्रदर्शित करता है।

$\dfrac{x-4}{5} = \dfrac{y-1}{2} = \dfrac{z-0}{1}$ ... (2)

(1) और (2) का प्रतिच्छेद बिन्दु प्राप्त करने के लिए —

$\dfrac{2r+1-4}{5} = \dfrac{3r+2-1}{2} = \dfrac{4r+3}{1} \Rightarrow \dfrac{2r-3}{5} = \dfrac{3r+1}{2} = \dfrac{4r+3}{1} \Rightarrow 4r - 6 = 15r + 5 \Rightarrow 11r = -11 \Rightarrow r = -1$

$\therefore$ (1) और (2) का प्रतिच्छेद बिन्दु $(-1, -1, -1)$ है।

$\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j})$ ... (1) $\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$ ... (2)

इनके प्रतिच्छेद बिन्दु के लिए

$3\lambda + 1 = 4 + 2\mu \Rightarrow 3\lambda - 2\mu - 3 = 0$ ... (3)

$1 - \lambda = 0 \Rightarrow \lambda = 1$ ... (4)

और $-1 = -1 + 3\mu$ $\mu = 0$

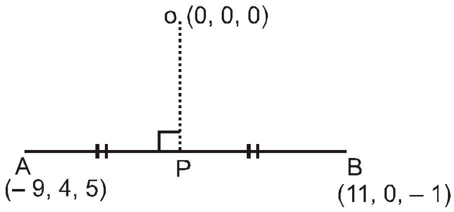
$\therefore$ प्रतिच्छेद बिन्दु $(4, 0, -1)$ है। $\therefore$ अभीष्ट दूरी $= \sqrt{(4+1)^2 + 1 + 0} = \sqrt{25+1} = \sqrt{26}$

C-5. Show that the foot of the perpendicular from the origin to the join of A(–9, 4, 5) and B (11, 0, –1) is the mid point of AB. Also find distance of point (2, 4, 4) from the line AB.

प्रदर्शित कीजिए कि बिन्दुओं A(–9, 4, 5) और B (11, 0, –1) को मिलाने वाली रेखा पर मूलबिन्दु से डाले गये लम्ब का पाद AB का मध्य बिन्दु है तथा रेखा AB से बिन्दु (2, 4, 4) की दूरी भी ज्ञात कीजिए।

Ans. 3

Sol.



$\because$ P (mid-point of AB) is (1, 2, 2)

$\because$ dr's of OP are : 1, 2, 2 and dr's of AB are : 20, – 4, – 6

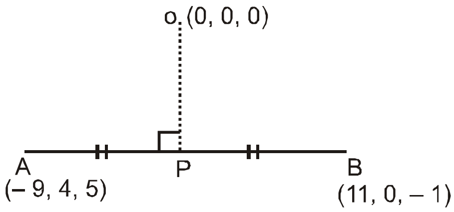
$\because$ $1 \times 20 + 2 \times (-4) + 2 \times (-6) = 20 - 12 - 8 = 0$

$\therefore$ OP $\perp$ AB

as the midpoint of (0, 0, 0) and (2, 4, 4) is (1, 2, 2) so the distance = $\sqrt{1+2^2+2^2} = 3$

Ans. 3

Hindi.



$\because$ P (AB का मध्यबिन्दु), (1, 2, 2) है $\because$ OP के दिक्अनुपात हैं 1, 2, 2

और AB के दिक्अनुपात हैं 20, – 4, – 6

$\because$ $1 \times 20 + 2 \times (-4) + 2 \times (-6) = 20 - 12 - 8 = 0$ $\therefore$ OP $\perp$ AB

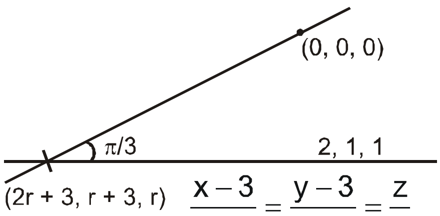
चूंकि (0, 0, 0) और (2, 4, 4) का मध्य बिन्दु (1, 2, 2) है इसलिए दूरी = $\sqrt{1+2^2+2^2} = 3$

C-6. Find the equation of the two lines through the origin which intersect the line $\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1}$ at an angle of $\pi/3$.

मूलबिन्दु से गुजरने वाली दो सरल रेखाओं के समीकरण ज्ञात कीजिए जो रेखा $\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1}$ को $\pi/3$ कोण पर काटती हैं।

Ans. $\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$, $\frac{x}{-1} = \frac{y}{1} = \frac{z}{-2}$

Sol.



$\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1}$

Any point on the given line is (2r + 3, r + 3, r)

Dr's of new line are $\equiv$ 2r + 3, r + 3, r

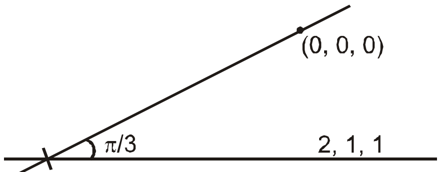
$$\cos\frac{\pi}{3} = \frac{2(2r+3)+1(r+3)+1.r}{\sqrt{6}\ \sqrt{(2r+3)^2+(r+3)^2+r^2}}$$

Solving above we get $r = -1, -2$

Dr's are $1, 2, -1$ or $-1, 1, -2$

equation of the line is $\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$ or $\frac{x}{-1} = \frac{y}{1} = \frac{z}{-2}$

Hindi.



(2r + 3, r + 3, r) $\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1}$

दी गई रेखा पर कोई बिन्दु है $(2r+3,\ r+3,\ r)$

नई रेखा के दिक्अनुपात हैं $\equiv 2r+3,\ r+3,\ r$

$$\cos\frac{\pi}{3} = \frac{2(2r+3)+1(r+3)+1.r}{\sqrt{6}\ \sqrt{(2r+3)^2+(r+3)^2+r^2}}$$

उपरोक्त को हल करने पर हमें प्राप्त होता है। $r = -1, -2$

दिक्अनुपात $1, 2, -1$ या $-1, 1, -2$ है।

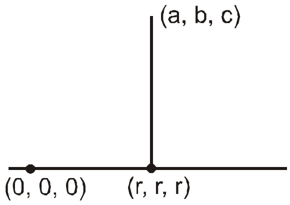
रेखा का समीकरण $\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$ या $\frac{x}{-1} = \frac{y}{1} = \frac{z}{-2}$ है।

C-7. The foot of the perpendicular from (a, b, c) on the line x = y = z is the point (r, r, r), then find the value of r.

बिन्दु (a, b, c) से रेखा x = y = z पर डाले गये लम्ब का पाद बिन्दु (r, r, r) है, तो r का मान ज्ञात कीजिए।

Ans. $r = \frac{a+b+c}{3}$

Sol.



$x = y = z = r$

$(r-a)r + (r-b)r + (r-c)r = 0$

$a + b + c = 3r$

C-8. Find the shortest distance between the lines :

$$\vec{r} = (4\hat{i}-\hat{j})+\lambda\ (\hat{i}+2\hat{j}-3\hat{k}) \text{ and } \vec{r} = (\hat{i}-\hat{j}+2\hat{k})+\mu\ (2\hat{i}+4\hat{j}-5\hat{k})$$

रेखाओं $\vec{r} = (4\hat{i}-\hat{j})+\lambda\ (\hat{i}+2\hat{j}-3\hat{k})$ और $\vec{r} = (\hat{i}-\hat{j}+2\hat{k})+\mu\ (2\hat{i}+4\hat{j}-5\hat{k})$ के बीच न्यूनतम दूरी ज्ञात कीजिए।

Ans. $\frac{6}{\sqrt{5}}$ unit इकाई

Sol. $SD = \frac{(\hat{i}-\hat{j}+2\hat{k}-4\hat{i}+\hat{j})\cdot[(\hat{i}+2\hat{j}-3\hat{k})\times(2\hat{i}+4\hat{j}-5\hat{k})]}{\left|(\hat{i}+2\hat{j}-3\hat{k})\times(2\hat{i}+4\hat{j}-5\hat{k})\right|} = \left|\frac{(-\ 3\hat{i}+2\hat{k})\ .\ (2\hat{i}-\hat{j})}{|2\hat{i}-\hat{j}|}\right| = \frac{6}{\sqrt{5}}$

C-9. Let L_1 and L_2 be the lines whose equation are $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$ respectively. A and B are two points on L_1 and L_2 respectively such that AB is perpendicular both the lines L_1 and L_2.Find points A, B and hence find shortest distance between lines L_1 and L_2

मानाकि रेखायें L_1 तथा L_2 जिनके समीकरण क्रमशः $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$ तथा $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$ हैं। A तथा B दो बिन्दु क्रमशः L_1 और L_2 पर इस प्रकार स्थित हैं कि AB दोनों रेखाओं L_1 तथा L_2 के लम्बवत् है। दोनों बिन्दु A व B ज्ञात कीजिए तथा L_1 व L_2 के मध्य न्यूनतम दूरी भी ज्ञात कीजिए। **[16JM120010]**

Ans. A = (3, 8, 3), B = (–3, – 7, 6), AB = $3\sqrt{30}$

Sol. Let the co-ordinates of A be $(3\lambda + 3, 8 - \lambda, \lambda + 3)$ and the co-ordinates of B be $(-3\mu - 3, 2\mu - 7, 4\mu + 6)$.

Then direction ratios of AB are $3\lambda + 3\mu + 6,\ -\lambda - 2\mu + 15,\ \lambda - 4\mu - 3$

$AB \perp L_1$ so $3(3\lambda + 3\mu + 6) - (-\lambda - 2\mu + 15) + (\lambda - 4\mu - 3) = 0$ i.e. $11\lambda + 7\mu = 0$

and $AB \perp L_2$ so $-3(3\lambda + 3\mu + 6) + 2(-\lambda - 2\mu + 15) + 4(\lambda - 4\mu - 3) = 0$ i.e. $-7\lambda - 29\mu = 0$

$\Rightarrow \lambda = \mu = 0$ so the point A is (3, 8, 3) and the point B is (–3, –7, 6)

$\therefore$ AB = $\sqrt{36+225+9} = \sqrt{270} = 3\sqrt{30}$

Hindi. मानाकि A के निर्देशांक $A(3\lambda + 3, 8 - \lambda, \lambda + 3)$ हैं तथा B के निर्देशांक $(-3\mu - 3, 2\mu - 7, 4\mu + 6)$ हैं।

तब AB के दिक्अनुपात $3\lambda + 3\mu + 6,\ -\lambda - 2\mu + 15,\ \lambda - 4\mu - 3$ होंगे

$AB \perp L_1$ इसलिए $3(3\lambda + 3\mu + 6) - (-\lambda - 2\mu + 15) + (\lambda - 4\mu - 3) = 0$ अर्थात् $11\lambda + 7\mu = 0$

तथा $AB \perp L_2$ इसलिए $-3(3\lambda + 3\mu + 6) + 2(-\lambda - 2\mu + 15) + 4(\lambda - 4\mu - 3) = 0$ अर्थात् $-7\lambda - 29\mu = 0$

$\Rightarrow \lambda = \mu = 0$

इसलिए बिन्दु A (3, 8, 3) है तथा बिन्दु B (–3, –7, 6) है।

$\therefore$ AB = $\sqrt{36+225+9} = \sqrt{270} = 3\sqrt{30}$

C-10. If $\vec{r} = (\hat{i}+2\hat{j}+3\hat{k})+\lambda\ (\hat{i}-\hat{j}+\hat{k})$ and $\vec{r} = (\hat{i}+2\hat{j}+3\hat{k})+\mu\ (\hat{i}+\hat{j}-\hat{k})$ are two lines, then find the equation of acute angle bisector of two lines.

यदि $\vec{r} = (\hat{i}+2\hat{j}+3\hat{k})+\lambda\ (\hat{i}-\hat{j}+\hat{k})$ और $\vec{r} = (\hat{i}+2\hat{j}+3\hat{k})+\mu\ (\hat{i}+\hat{j}-\hat{k})$ दो रेखाएँ हैं, तो दोनों रेखाओं के न्यूनकोण समद्विभाजक का समीकरण ज्ञात कीजिए।

Ans. $\vec{r} = (\hat{i}+2\hat{j}+3\hat{k})+t\ (\hat{j}-\hat{k})$, where t is parameter (जहाँ t प्राचल है।)

Sol. Angle between two vectors = $\frac{1\times1+(-1)(1)+(1)(-1)}{\sqrt{3}\ \sqrt{3}} = -\frac{1}{3}$. Hence obtuse angle between them.

Vector along acute angle bisector = $\lambda\left[\frac{\hat{i}-\hat{j}+\hat{k}}{\sqrt{3}}-\frac{\hat{i}+\hat{j}-\hat{k}}{\sqrt{3}}\right]$; $\frac{2\lambda}{\sqrt{3}}\left[-\hat{j}+\hat{k}\right]=t(\hat{j}-\hat{k})$

hence equation of acute angle bisector = $(\hat{i}+2\hat{j}+3\hat{k})+t(\hat{j}-\hat{k})$

Hindi दो सदिशों के बीच कोण = $\frac{1\times1+(-1)(1)+(1)(-1)}{\sqrt{3}\ \sqrt{3}} = -\frac{1}{3}$. अतः इनके मध्य अधिक कोण है

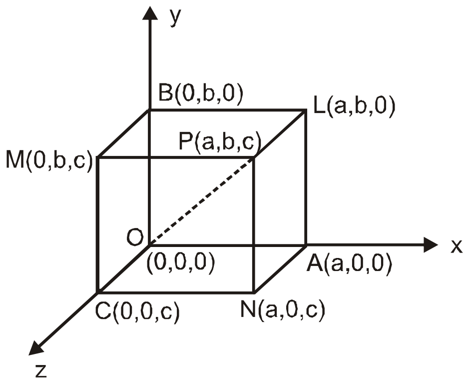
न्यून कोण समद्विभाजक के अनुदिश सदिश है $=\lambda\left[\frac{\hat{i}-\hat{j}+\hat{k}}{\sqrt{3}}-\frac{\hat{i}+\hat{j}-\hat{k}}{\sqrt{3}}\right]$; $\frac{2\lambda}{\sqrt{3}}\left[-\hat{j}+\hat{k}\right]=t(\hat{j}-\hat{k})$

अतः न्यून कोण समद्विभाजक की समीकरण है = $(\hat{i}+2\hat{j}+3\hat{k})+t(\hat{j}-\hat{k})$

C-11. The edges of a rectangular parallelopiped are a, b, c; show that the angles between two of the four diagonals are given by $\cos^{-1}\left(\pm\left(\frac{a^2+b^2-c^2}{a^2+b^2+c^2}\right)\right)$ or $\cos^{-1}\left(\pm\left(\frac{a^2-b^2+c^2}{a^2+b^2+c^2}\right)\right)$ or $\cos^{-1}\left(\pm\left(\frac{a^2-b^2-c^2}{a^2+b^2+c^2}\right)\right)$.

एक आयताकार समान्तरषट्फलक के किनारे a, b, c है, प्रदर्शित कीजिए कि चारों विकर्णों के मध्य कोण $\cos^{-1}\left(\pm\left(\frac{a^2+b^2-c^2}{a^2+b^2+c^2}\right)\right)$ or $\cos^{-1}\left(\pm\left(\frac{a^2-b^2+c^2}{a^2+b^2+c^2}\right)\right)$ or $\cos^{-1}\left(\pm\left(\frac{a^2-b^2-c^2}{a^2+b^2+c^2}\right)\right)$. होता है।

Sol.



Dr's of OP are a, b, c ; Dr's of AM are –a, b,c

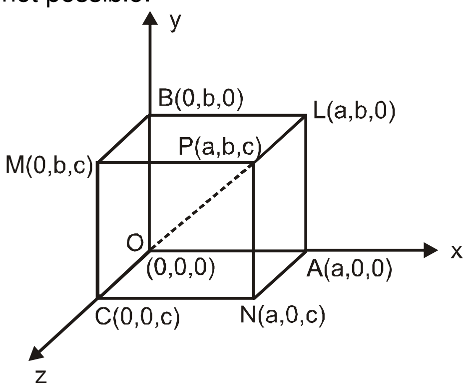
If θ be the angle between OP and AM, we get $\cos\theta = \dfrac{a(-a)+b(b)+c(c)}{\sqrt{a^2+b^2+c^2}\sqrt{a^2+b^2+c^2}} = \dfrac{-a^2+b^2+c^2}{a^2+b^2+c^2}$

Similarly we can find the angle between other pairs and hence angles in general between six pairs are given by

$$\theta = \cos^{-1}\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right)$$

Note : All +ve or -ve signs can not be taken because angle between diagonals are 0 or π radian is not possible.

Hindi



OP के दिक् अनुपात a, b,c हैं। AM के दिक् अनुपात – a, b,c हैं।

यदि θ , OP तथा AM के मध्य कोण है, तो

$$\cos\theta = \frac{a(-a)+b(b)+c(c)}{\sqrt{a^2+b^2+c^2}\sqrt{a^2+b^2+c^2}} = \frac{-a^2+b^2+c^2}{a^2+b^2+c^2}$$

इसी तरह हम दूसरे युग्मों के मध्य कोण ज्ञात कर सकते है।

इस प्रकार सामान्य रूप में छः युग्मों के मध्य कोण $\theta = \cos^{-1}\left(\dfrac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right)$ द्वारा दिया जाता है

नोट : एक साथ सभी धनात्मक या सभी ऋणात्मक चिन्ह नहीं लिये जा सकते है क्योंकि विकर्णो के मध्य कोण 0 या π रेडियन संभव नहीं है।

C-12. Show that equation of angle bisectors of line $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{b} + \mu\vec{a}$ are $\vec{r} = (\vec{a}+\vec{b}) + \gamma\left(|\vec{b}|\vec{a} \pm |\vec{a}|\vec{b}\right)$

दर्शाइयें कि रेखाओं $\vec{r} = \vec{a} + \lambda\vec{b}$ और $\vec{r} = \vec{b} + \mu\vec{a}$ के कोण अर्द्धकों के समीकरण $\vec{r} = (\vec{a}+\vec{b}) + \gamma\left(|\vec{b}|\vec{a} \pm |\vec{a}|\vec{b}\right)$ है।

Sol. Point of intersection of $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{b} + \mu\vec{a}$ is $\vec{a} + \vec{b}$

Now angle bisector of vectors $\vec{a}$ & $\vec{b}$ is $\vec{a}+\vec{b}$ and angle bisector of vector $\vec{a}$ and $-\vec{b}$ is $\vec{a} - \vec{b}$ so equation of angle bisectors of given lines are $\vec{r} = \left(\vec{a}+\vec{b}\right) + \gamma\left(\hat{a} \pm \hat{b}\right)$

Hindi : $\vec{r} = \vec{a} + \lambda\vec{b}$ और $\vec{r} = \vec{b} + \mu\vec{a}$ का प्रतिच्छेद बिन्दु $\vec{a} + \vec{b}$ है।
अब व के मध्य कोण अर्द्धक तथा $\vec{a}$ व $-\vec{b}$ के मध्य कोण अर्द्धक $\vec{a} - \vec{b}$ है। इसलिए दी गई रेखाओं के कोण अर्द्धकों के समीकरण $\vec{r} = \left(\vec{a}+\vec{b}\right) + \gamma\left(\hat{a} \pm \hat{b}\right)$ है।

C-13. Prove that the shortest distance between the diagonals of a rectangular parallelepiped whose coterminous sides are a, b, c and the edges not meeting it are $\frac{bc}{\sqrt{b^2+c^2}}, \frac{ca}{\sqrt{c^2+a^2}}, \frac{ab}{\sqrt{a^2+b^2}}$
सिद्ध कीजिए आयताकार समान्तर षट्फलक के विकर्णों के मध्य लघुत्तम दूरी जिसकी आसन्न भुजाएं a, b, c है तथा यह किनारों को नहीं मिल रही है $\frac{bc}{\sqrt{b^2+c^2}}, \frac{ca}{\sqrt{c^2+a^2}}, \frac{ab}{\sqrt{a^2+b^2}}$ है।

Sol. Let coordinates of rectangular parallelepiped are A(0,0,0); B(a,0,0); C(a,b,0); D(0,b,0); E(0,0,c); F(a,0,c); G(a,b,c); H(0,b,c). Then equation of diagonal AG is $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$ and equation of BC is
$\frac{x-a}{0} = \frac{y-0}{1} = \frac{z-0}{0}$

$\Rightarrow$ shortest distance between AG and BC = $\frac{(a\hat{i}).(\hat{j}\times(a\hat{i}+b\hat{j}+c\hat{k}))}{|\hat{j}\times(a\hat{i}+b\hat{j}+c\hat{k}|} = \frac{ac}{\sqrt{a^2+c^2}}$

Similarly shortest distance between AG and DC = $\frac{bc}{\sqrt{b^2+c^2}}$

Similarly shortest distance between AG and BF = $\frac{ba}{\sqrt{b^2+a^2}}$

Sol. माना समान्तर षट्फलक के निर्देशांक A(0,0,0); B(a,0,0); C(a,b,0); D(0,b,0); E(0,0,c); F(a,0,c); G(a,b,c); H(0,b,c) तब विकर्ण AG का समीकरण $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$ AG और BC के मध्य दुरी $\frac{x-a}{0} = \frac{y-0}{1} = \frac{z-0}{0}$

$\Rightarrow$ AG तथा BC के मध्य दुरी $= \frac{(a\hat{i}).(\hat{j}\times(a\hat{i}+b\hat{j}+c\hat{k}))}{|\hat{j}\times(a\hat{i}+b\hat{j}+c\hat{k}|} = \frac{ac}{\sqrt{a^2+c^2}}$

इसीप्रकार AG तथा DC के मध्य दुरी = $\frac{bc}{\sqrt{b^2+c^2}}$

इसीप्रकार AG तथा BF के मध्य लघुत्तम दुरी = $\frac{ba}{\sqrt{b^2+a^2}}$

Section (D) : STP, VTP, Vector equation, LI/LD
खण्ड (D) : अदिश त्रिकगुणन,सदिश त्रिकगुणन, सदिश समीकरण, रेखिय परतंत्र/रेखिय स्वतंत्र

D-1. Show that दर्शाइये कि $\{(\vec{a} + \vec{b} + \vec{c}) \times (\vec{c} - \vec{b})\} . \vec{a} = 2\left[\vec{a}\ \vec{b}\ \vec{c}\right]$.

Sol. Required अभीष्ट $= (\vec{a} \times \vec{c} - \vec{a} \times \vec{b} + \vec{b} \times \vec{c} - \vec{c} \times \vec{b}) . \vec{a} = \left[\vec{b}\ \vec{c}\ \vec{a}\right] - \left[\vec{c}\ \vec{b}\ \vec{a}\right]\left[\vec{a}\ \vec{b}\ \vec{c}\right]$
$= 2\left[\vec{a}\ \vec{b}\ \vec{c}\right]$

D-2. Given unit vectors $\hat{m}$, $\hat{n}$ and $\hat{p}$ such that $\left(\hat{m} \hat{\ } \hat{n}\right) = \hat{p}\ \hat{\ }(\hat{m} \times \hat{n}) = \alpha$, then find value of $\left[\hat{n}\ \ \hat{p}\ \ \hat{m}\right]$ in terms of α.

इकाई सदिश $\hat{m}$, $\hat{n}$ एवं $\hat{p}$ इस प्रकार है कि $(\hat{m} \wedge \hat{n}) = \hat{p} \wedge (\hat{m} \times \hat{n}) = \alpha$, तो $[\hat{n}\ \hat{p}\ \hat{m}]$ का मान α के पदों में ज्ञात कीजिए।

Ans. $\sin\alpha \cos\alpha$

Sol. $[\hat{n}\ \hat{p}\ \hat{m}] = [\hat{p}\ \hat{m}\ \hat{n}] = \hat{p}.(\hat{m}\times\hat{n}) = |\hat{p}|.|\hat{m}\times\hat{n}|\ cos a = |\hat{m} \times \hat{n}|\ cos a = sin a\ cos a$

D-3. Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$ be three non-zero vectors such that $\vec{c}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. If the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$, then $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2$ is equal to:

माना $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ और $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$ तीन अशून्य सदिश इस तरह है कि $\vec{c}$ एक इकाई सदिश है जो $\vec{a}$ तथा $\vec{b}$ दोनों के लम्बवत् है। यदि $\vec{a}$ तथा $\vec{b}$ के मध्य कोण $\frac{\pi}{6}$ है, तो $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2$ बराबर है–

Ans. $\frac{1}{4}(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)$

Sol. $$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2 = [\vec{a}\ \ \vec{b}\ \ \vec{c}]^2 = ((\vec{a}\times\vec{b}).\vec{c})^2 = (ab\sin\theta\vec{c}.\vec{c})^2 = \frac{a^2b^2}{4}$$

$$= \frac{1}{4}(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)$$

D-4. Examine for coplanarity of the following sets of points
निम्न बिन्दुओं के समुच्चयों की समतलता की जाँच कीजिए –

(a) $4\hat{i} + 8\hat{j} + 12\hat{k}$, $2\hat{i} + 4\hat{j} + 6\hat{k}$, $3\hat{i} + 5\hat{j} + 4\hat{k}$, $5\hat{i} + 8\hat{j} + 5\hat{k}$.

(b) $3\vec{a} + 2\vec{b} - 5\vec{c}$, $3\vec{a} + 8\vec{b} + 5\vec{c}$, $-3\vec{a} + 2\vec{b} + \vec{c}$, $\vec{a} + 4\vec{b} - 3\vec{c}$. Where $\vec{a}, \vec{b}, \vec{c}$ are noncoplanar
$3\vec{a} + 2\vec{b} - 5\vec{c}$, $3\vec{a} + 8\vec{b} + 5\vec{c}$, $-3\vec{a} + 2\vec{b} + \vec{c}$, $\vec{a} + 4\vec{b} - 3\vec{c}$ जहाँ $\vec{a}, \vec{b}, \vec{c}$ असमतलीय है।

Ans. (a) Coplanar (b) Non-coplanar (a) समतलीय (b) असमतलीय

Sol. **(a)** Let $A \equiv 4\hat{i}+8\hat{j}+12\hat{k}$; $B \equiv 2\hat{i}+4\hat{j}+6\hat{k}$; $C \equiv 3\hat{i}+5\hat{j}+4\hat{k}$; $D \equiv 5\hat{i}+8\hat{j}+5\hat{k}$

$\overrightarrow{AB} = -\ 2\hat{i}-4\hat{j}-6\hat{k}$; $\overrightarrow{BC} = \hat{i}-\hat{j}-2\hat{k}$; $\overrightarrow{CD} = 2\hat{i}+3\hat{j}+\hat{k}$

$$[\overrightarrow{AB}\ \ \overrightarrow{BC}\ \ \overrightarrow{CD}] = \begin{vmatrix} -2 & -4 & -6 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix} = [-2(1+6) + 4(1+4) - 6(3-2)] = -14 + 20 - 6 = 0$$

Hence coplanar

(b) Using these 4 points let us create any 3 vectors $\vec{v}_1 = 6\vec{b}+10\vec{c}$, $\vec{v}_2 = -6a+6\vec{c}$
$\vec{v}_3 = -2\vec{a}+2\vec{b}+2\vec{c}$ check coplanarity of these vectors

$$[\vec{v}_1\ \ \vec{v}_2\ \ \vec{v}_3] = \begin{vmatrix} 0 & 6 & 10 \\ -6 & 0 & 6 \\ -2 & 2 & 2 \end{vmatrix}[\vec{a}\ \ \vec{b}\ \ \vec{c}] \neq 0$$ hence non coplanar

Hindi. (a) माना $A \equiv 4\hat{i}+8\hat{j}+12\hat{k}$; $B \equiv 2\hat{i}+4\hat{j}+6\hat{k}$; $C \equiv 3\hat{i}+5\hat{j}+4\hat{k}$; $D \equiv 5\hat{i}+8\hat{j}+5\hat{k}$

$\overrightarrow{AB} = -\ 2\hat{i}-4\hat{j}-6\hat{k}$; $\overrightarrow{BC} = \hat{i}-\hat{j}-2\hat{k}$; $\overrightarrow{CD} = 2\hat{i}+3\hat{j}+\hat{k}$

$$[\overrightarrow{AB}\ \ \overrightarrow{BC}\ \ \overrightarrow{CD}] = \begin{vmatrix} -2 & -4 & -6 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix} = [-2(1+6)+4(1+4)-6(3-2)] = -14+20-6 = 0$$

अतः समतलीय है

(b) इन 4 बिन्दुओं का उपयोग करके 3 सदिश बनाने पर $\vec{v}_1 = 6\vec{b}+10\vec{c}$, $\vec{v}_2 = -6a+6\vec{c}$

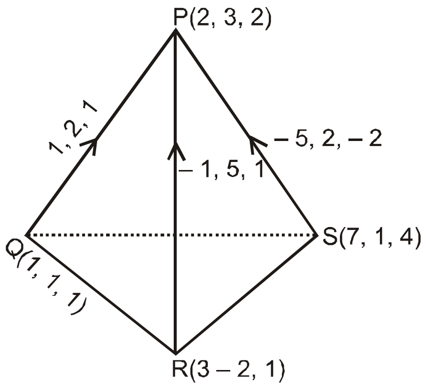
$\vec{v}_3 = -2\vec{a}+2\vec{b}+2\vec{c}$ इनकी समतलता की जाँच करने पर

$$[\vec{v}_1\ \ \vec{v}_2\ \ \vec{v}_3] = \begin{vmatrix} 0 & 6 & 10 \\ -6 & 0 & 6 \\ -2 & 2 & 2 \end{vmatrix} [\vec{a}\ \ \vec{b}\ \ \vec{c}] \neq 0$$ अतः असमतलीय है।

D-5. The vertices of a tetrahedron are P(2, 3, 2), Q(1, 1, 1), R(3, –2, 1) and S (7, 1, 4).
(i) Find the volume of tetrahedron
(ii) Find the shortest distance between the lines PQ & RS.
चतुष्फलक के शीर्ष P(2, 3, 2), Q(1, 1, 1), R(3, –2, 1) और S (7, 1, 4) है तब ज्ञात कीजिये
(i) चतुष्फलक का आयतन
(ii) रेखा PQ तथा RS के मध्य लघुत्तम दूरी

Ans. (i) 1/2 unit³ (ii) $\dfrac{3}{\sqrt{35}}$ unit

Sol.



(i) $$v = \left| \frac{1}{6} \begin{vmatrix} 1 & 2 & 1 \\ -1 & 5 & 1 \\ -5 & 2 & -2 \end{vmatrix} \right|$$

$$= \left| \frac{1}{6}\ \{1\ \ (-10-2)-2(2+5)+1\ \ (-2+25)\} \right|$$

$$= \left| \frac{1}{6}(-12-14+23) \right| = \frac{1}{2}$$

(ii) $$(\hat{i}-5\hat{j}-\hat{k}).\frac{(3\hat{i}+\hat{j}-5\hat{k})}{\sqrt{9+1+25}} = \frac{3}{\sqrt{35}}$$

D-6. Are the following set of vectors linearly independent?
क्या निम्नलिखित सदिशों के समुच्चय एकघाततः स्वतंत्र हैं ?

(i) $\vec{a} = \hat{i} - 2\ \hat{j} + 3\hat{k}$, $\vec{b} = 3\ \hat{i} - 6\ \hat{j} + 9\hat{k}$

(ii) $\vec{a} = -2\hat{i} - 4\hat{k}$, $\vec{b} = \hat{i} - 2\hat{j} - \hat{k}$, $\vec{c} = \hat{i} - 4\hat{j} + 3\hat{k}$

Ans. (i) No नहीं (ii) Yes हाँ

Sol. (i) $\vec{b} = 3\vec{a}$ linearly dependent (एक घातत: परतंत्र)

(iii) $[\vec{a}\ \ \vec{b}\ \ \vec{c}] \neq 0$ linearly independent (एक घातत: स्वतन्त्र)

D-7. Find value of $x \in R$ for which the vectors $\vec{a} = (1, -2, 3)$, $\vec{b} = (-2, 3, -4)$, $\vec{c} = (1, -1, x)$ form a linearly dependent system.

x के ऐसे वास्तविक मान ज्ञात कीजिए जिनके लिए सदिश $\vec{a} = (1, -2, 3)$, $\vec{b} = (-2, 3, -4)$, $\vec{c} = (1, -1, x)$ एक घातत: परतन्त्र निकाय बनाते है।

Ans. $x = 1$

Sol. For linearly dependent vectors (एक घातत: परतन्त्र सदिशों के लियें)

$\ell(i - 2j + 3k) + m(-2i + 3j - 4k) + n(i - j + xk) = 0 \Rightarrow \ell - 2m + n = 0,\ -2\ell + 3m - n = 0$

$3\ell - 4m + nx = 0$ $\quad \therefore \quad \begin{vmatrix} 1 & -2 & 1 \\ -2 & 3 & -1 \\ 3 & -4 & x \end{vmatrix} = 0$ is $x = 1$

D-8. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors and $\vec{v}\cdot\vec{a} = \vec{v}\cdot\vec{b} = \vec{v}\cdot\vec{c} = 0$, then find value of $\vec{v}$.

यदि $\vec{a}, \vec{b}, \vec{c}$ असमतलीय सदिश है और $\vec{v}\cdot\vec{a} = \vec{v}\cdot\vec{b} = \vec{v}\cdot\vec{c} = 0$ हो, तो $\vec{v}$ का मान ज्ञात कीजिए।

Ans. $\vec{v} = 0$

Sol. Let $\vec{v} = \lambda_1\vec{a} + \lambda_2\vec{b} + \lambda_3\vec{c}$ $\quad [\vec{a},\ \vec{b},\ \vec{c}$ are non coplanar]

$\vec{v}.\vec{a} = \vec{v}.\vec{b} = \vec{v}.\vec{c} = 0 \quad \Rightarrow \quad \lambda_1\vec{a}.\vec{a} + \lambda_2\vec{a}.\vec{b} + \lambda_3\vec{a}.\vec{c} = 0$

$\lambda_1\vec{b}.\vec{a} + \lambda_2\vec{b}.\vec{b} + \lambda_3\vec{b}.\vec{c} = 0 \quad \Rightarrow \quad \lambda_1\vec{c}.\vec{a} + \lambda_2\vec{c}.\vec{b} + \lambda_3\vec{c}.\vec{c} = 0$

Only possible values of $\lambda_1, \lambda_2, \lambda_3 = 0$ as $\begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix} \neq 0$

Hindi माना $\vec{v} = \lambda_1\vec{a} + \lambda_2\vec{b} + \lambda_3\vec{c}$ $\quad [\vec{a},\ \vec{b},\ \vec{c}$ असमतलीय है]

$\vec{v}.\vec{a} = \vec{v}.\vec{b} = \vec{v}.\vec{c} = 0 \quad \Rightarrow \quad \lambda_1\vec{a}.\vec{a} + \lambda_2\vec{a}.\vec{b} + \lambda_3\vec{a}.\vec{c} = 0$

$\lambda_1\vec{b}.\vec{a} + \lambda_2\vec{b}.\vec{b} + \lambda_3\vec{b}.\vec{c} = 0 \quad \Rightarrow \quad \lambda_1\vec{c}.\vec{a} + \lambda_2\vec{c}.\vec{b} + \lambda_3\vec{c}.\vec{c} = 0$

$\lambda_1, \lambda_2, \lambda_3$ का सम्भव मान शून्य होगा क्योंकि $\begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix} \neq 0$

D-9. Let $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{c} = 3\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{d} = 3\hat{i} - \hat{j} - 2\hat{k}$, then

(i) if $\vec{a} \times (\vec{b} \times \vec{c}) = p\vec{a} + q\vec{b} + r\vec{c}$, then find value of p, q and r.

(ii) find the value of $(\vec{a} \times \vec{b}) \times (\vec{a} \times \vec{c}).\vec{d}$

मानाकि $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{c} = 3\hat{i} + 2\hat{j} + \hat{k}$ और $\vec{d} = 3\hat{i} - \hat{j} - 2\hat{k}$ है, तो

(i) यदि $\vec{a} \times (\vec{b} \times \vec{c}) = p\vec{a} + q\vec{b} + r\vec{c}$ हो, तो p, q और r के मान ज्ञात कीजिए।

(ii) $(\vec{a} \times \vec{b}) \times (\vec{a} \times \vec{c}).\vec{d}$ का मान ज्ञात कीजिए।

Ans. (i) $p = 0; q = 10; r = -3$ (ii) -100

Sol. (i) $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$ $\Rightarrow$ $10\ \vec{b}-3\ \vec{c} = p\vec{a}+q\vec{b}+r\vec{c}$

$p = 0, q = +10,\ r = -3$ $[\vec{a},\vec{b},\vec{c}$ are non coplanar]

(ii) $(\vec{a}\times\vec{b}) \times (\vec{a}\times\vec{c}) . \vec{d} = \{\left((\vec{a}\times\vec{b}).\vec{c}\right)\vec{a} - \left((\vec{a}\times\vec{b}).\vec{a}\right)\vec{c}\}\cdot \vec{d} = [\vec{a}\ \ \vec{b}\ \ \vec{c}]\ \vec{a}.\vec{d} - 0 = 20 \times (-5) = -100$

Hindi (i) $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$ $\Rightarrow$ $10\vec{b}-3\vec{c} = p\vec{a}+q\vec{b}+r\vec{c}$

$p = 0, q = +10,\ r = -3$ $[\vec{a},\vec{b},\vec{c}$ असमतलीय हैं]

(ii) $(\vec{a}\times\vec{b}) \times (\vec{a}\times\vec{c}) . \vec{d} = \{\left((\vec{a}\times\vec{b}).\vec{c}\right)\vec{a} - \left((\vec{a}\times\vec{b}).\vec{a}\right)\vec{c}\}\cdot \vec{d} = [\vec{a}\ \ \vec{b}\ \ \vec{c}]\ \vec{a}.\vec{d} - 0 = 20 \times (-5) = -100$

D-10. Given that $\vec{x}+\frac{1}{\vec{p}^2}(\vec{p}\ .\ \vec{x})\ \vec{p}=\vec{q}$, then show that $\vec{p}.\vec{x}=\frac{1}{2}(\vec{p}.\vec{q})$ and hence find $\vec{x}$ in terms of $\vec{p}$ and $\vec{q}$.

दिया गया है कि $\vec{x}+\frac{1}{\vec{p}^2}(\vec{p}\ .\ \vec{x})\ \vec{p}=\vec{q}$, तो प्रदर्शित कीजिए कि $\vec{p}.\vec{x}=\frac{1}{2}(\vec{p}.\vec{q})$ और $\vec{x}$ के मान को $\vec{p}$ और $\vec{q}$ के पदों में ज्ञात कीजिए।

Ans. $\vec{x}=\vec{q}-\frac{(\vec{p}\ .\ \vec{q})\ \vec{p}}{2|\vec{p}|^2}$

Sol. Dot with $\vec{p}$ & obtain answer. $\vec{p}$ के साथ अदिश गुणनफलन करने पर उत्तर प्राप्त होता है।

$\vec{x}+\frac{1}{|p|^2}\frac{1}{2}(p.q)\ \ \vec{p}=\vec{q}$ $\Rightarrow$ $\vec{x}=\vec{q}-\frac{(\vec{p}\ .\ \vec{q})\ \vec{p}}{2|\vec{p}|^2}$

D-11. Let there exist a vector $\vec{x}$ satisfying the conditions $\vec{x} \times \vec{a} = \vec{c}\times\vec{d}$ and $\vec{x} + 2\vec{d} = \left(\vec{v}\times\vec{d}\right)$. Find $\vec{x}$ in terms of $\vec{a}$, $\vec{c}$ and $\vec{d}$

माना सदिश $\vec{x}$ इस प्रकार अस्तित्व है कि प्रतिबन्ध $\vec{x} \times \vec{a} = \vec{c}\times\vec{d}$ और $\vec{x} + 2\vec{d} = \left(\vec{v}\times\vec{d}\right)$ को संतुष्ट करता है। $\vec{a}$, $\vec{c}$ और $\vec{d}$ को $\vec{x}$ के पदों में ज्ञात कीजिए।

Ans. $\vec{x} = \frac{\vec{d}\times(\vec{c}\times\vec{d})-2|\vec{d}|^2\,\vec{a}}{\vec{d}.\vec{a}}$

Sol. $\vec{d} \times \left(\vec{x}\times\vec{a}\right) = \vec{d} \times \left(\vec{c}\times\vec{d}\right)$

$\Rightarrow \left(\vec{d}.\vec{a}\right)\vec{x} - \left(\vec{d}.\vec{x}\right)\vec{a} = \vec{d}\times\left(\vec{c}\times\vec{d}\right)$ ....(i)

Now $\vec{d}.\vec{x} + 2\vec{d}.\vec{d} = \vec{d}.\left(\vec{v}\times\vec{d}\right) = 0$

$\Rightarrow \vec{d}.\vec{x} = -2\left|\vec{d}\right|^2$ ....(ii)

From (i) and (ii) we get $\vec{x} = \frac{\vec{d}\times(\vec{c}\times\vec{d})-2\left|\vec{d}\right|^2\vec{a}}{\vec{d}.\vec{a}}$ ((i) और (ii) से)

Section (E) : Plane
खण्ड (E) : समतल

E-1. Find equation of plane

(i) Which passes through (0, 1, 0), (0, 0, 1), (1, 2, 3)

(ii) Which passes through (0, 1, 0) and contains two vectors $\hat{i}+\hat{j}-\hat{k}$ & $2\hat{i}-\hat{j}$.

(iii) Whose normal is $\hat{i}+\hat{j}+\hat{k}$ & which passes through (1, 2, 1).

(iv) Which makes equal intercepts on co-ordinate axis and passes through (1, 2, 3)

समतल का समीकरण ज्ञात कीजिए

(i) जो बिन्दुओं (0, 1, 0), (0, 0, 1), (1, 2, 3) से गुजरता है।

(ii) जो बिन्दुओं (0, 1, 0) से गुजरता है तथा $\hat{i}+\hat{j}-\hat{k}$ तथा $2\hat{i}-\hat{j}$ को समाहित करता है

(iii) जिसका अभिलम्ब $\hat{i}+\hat{j}+\hat{k}$ है जो बिन्दु (1, 2, 1) से गुजरता है।

(iv) जो निर्देशांक अक्षों पर बराबर अन्तखण्ड काटती है तथा (1, 2, 3) से गुजरती है।

Ans. (i) $4x - y - z + 1 = 0$ (ii) $x + 2y + 3z - 2 = 0$ (iii) $x + y + z - 4 = 0$ (iv) $x + y + z - 6 = 0$

Sol. (i) $\begin{vmatrix} x & y-1 & z \\ 0 & 1 & -1 \\ -1 & -1 & -3 \end{vmatrix} = 0 \Rightarrow x(-3-1) - (y-1)(0-1) + z(1) = 0$

$-4x + y - 1 + z = 0 \Rightarrow 4x - y - z + 1 = 0$

(ii) $\begin{vmatrix} x & y-1 & z \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{vmatrix} = 0 \Rightarrow x(-1) - (y-1)(2) + z(-1-2) = 0$

$-x - 2y + 2 - 3z = 0 \Rightarrow x + 2y + 3z - 2 = 0$

(iii) $x + y + z = 1 + 2 + 1 \Rightarrow x + y + z - 4 = 0$

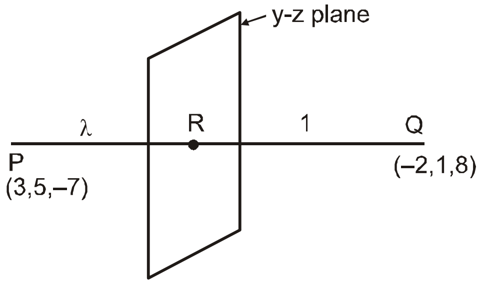
(iv) $x + y + z = 1 + 2 + 3 \Rightarrow x + y + z - 6 = 0$

E-2. Find the ratio in which the line joining the points (3, 5,–7) and (–2, 1, 8) is divided by the yz-plane. Find also the point of intersection of the plane and the line **[16JM120011]**

बिन्दु (3, 5,–7) और (–2, 1, 8) को जोड़ने वाली रेखा को yz-समतल किस अनुपात में विभाजित करता है, ज्ञात कीजिए साथ साथ ही समतल और रेखा का प्रतिच्छेद बिन्दु भी ज्ञात कीजिए।

Ans. 3 : 2, (0,13/5, 2)

Sol.



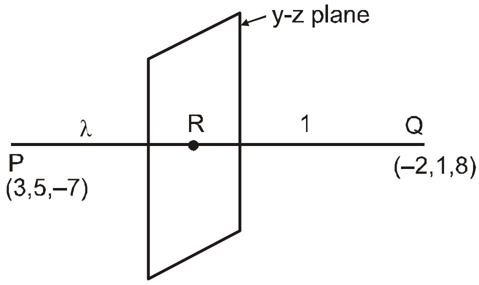
$\therefore \quad R\left(\frac{-2\lambda+3}{\lambda+1}, \frac{\lambda+5}{\lambda+1}, \frac{8\lambda-7}{\lambda+1}\right)$

$\because$ R lies on y –z plane

$\therefore \quad \frac{-2\lambda+3}{\lambda+1} = 0 \Rightarrow \lambda = \frac{3}{2}$

the ratio is 3 : 2 $\quad \therefore R\left(0, \frac{13}{5}, 2\right)$

Hindi



$\therefore \quad R\left(\frac{-2\lambda+3}{\lambda+1}, \frac{\lambda+5}{\lambda+1}, \frac{8\lambda-7}{\lambda+1}\right)$

$\because$ बिन्दु R समतल yz पर स्थित है

$\therefore \quad \frac{-2\lambda+3}{\lambda+1} = 0 \quad \Rightarrow \lambda = \frac{3}{2}$

अभीष्ट अनुपात 3 : 2 है $\quad \therefore \quad R\left(0, \frac{13}{5}, 2\right)$

E-3. Find the locus of the point whose sum of the square of distances from the planes $x + y + z = 0$, $x - z = 0$ and $x - 2y + z = 0$ is 9

उस बिन्दु का बिन्दुपथ ज्ञात कीजिए, जिसकी समतलों $x + y + z = 0$, $x - z = 0$ तथा $x - 2y + z = 0$ से दूरियों के वर्गो का योग 9 हो।

Ans. $x^2 + y^2 + z^2 = 9$

Sol. Let point is (α, β, γ) माना कोई बिन्दु (α, β, γ) है, तो

$$\left|\frac{\alpha+\beta+\gamma}{\sqrt{3}}\right|^2 + \left|\frac{\alpha-\gamma}{\sqrt{2}}\right|^2 + \left|\frac{\alpha-2\beta+\gamma}{\sqrt{6}}\right|^2 = 9 \quad \Rightarrow \quad 2(\alpha+\beta+\gamma)^2 + 3(\alpha-\gamma)^2 + (\alpha-2\beta+\gamma)^2 = 54$$

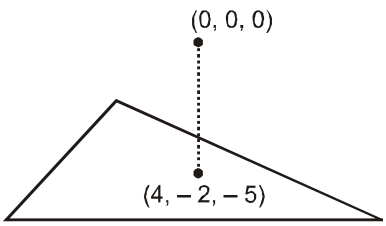
$6\alpha^2 + 6\beta^2 + 6\gamma^2 = 54 \quad \Rightarrow \quad$ Locus is $x^2 + y^2 + z^2 = 9$

$\Rightarrow$ बिन्दुपथ $x^2 + y^2 + z^2 = 9$ है।

E-4. The foot of the perpendicular drawn from the origin to the plane is (4, –2, –5), then find the vector equation of plane.

मूलबिन्दु से एक समतल पर डाले गये लम्ब का पाद (4, –2, –5) है, तो समतल का सदिश समीकरण ज्ञात कीजिए।

Ans. $\vec{r} \cdot (4\hat{i} - 2\hat{j} - 5\hat{k}) = 45$



Sol. Equation of plane is समतल का समीकरण है।

$\left\{\vec{r} - (4\hat{i} - 2\hat{j} - 5\hat{k})\right\} \cdot \left\{4\hat{i} - 2\hat{j} - 5\hat{k}\right\} = 0 \quad \Rightarrow \quad \vec{r} \cdot \left\{4\hat{i} - 2\hat{j} - 5\hat{k}\right\} = 45$

E-5. Let P (1, 3, 5) and Q(–2, 1, 4) be two points from which perpendiculars PM and QN are drawn to the x-z plane. Find the angle that the line MN makes with the plane $x + y + z = 5$.

माना कि P (1, 3, 5) तथा Q(–2, 1, 4) दो बिन्दु है जिनसे x-z समतल पर क्रमशः PM तथा QN लम्ब डाले जाते हैं। रेखा MN द्वारा समतल $x + y + z = 5$ के साथ बनाया गया कोण ज्ञात कीजिए।

Ans. $\sin^{-1} \frac{4}{\sqrt{30}}$

Sol. P(1, 3, 5) ; Q (–2, 1, 4) ; M(1, 0, 5) ; N(–2, 0, 4)

$\overrightarrow{NM} = (3\hat{i} + \hat{k}) \quad \sin\theta = \left|\frac{\overrightarrow{MN} \cdot (\hat{i}+\hat{j}+\hat{k})}{|MN|\sqrt{3}}\right| = \frac{3+1}{\sqrt{10}\sqrt{3}} = \frac{4}{\sqrt{30}}$

$\therefore$ Angle b/w plane and line MN is $\sin^{-1} \frac{4}{\sqrt{30}}$

Hindi. P(1, 3, 5) ; Q (–2, 1, 4) ; M(1, 0, 5) ; N(–2, 0, 4)

$\overrightarrow{NM} = (3\hat{i}+\hat{k})$ $\quad \sin\theta = \left|\dfrac{\overrightarrow{MN}\ .\ (\hat{i}+\hat{j}+\hat{k})}{|MN|\sqrt{3}}\right| = \dfrac{3+1}{\sqrt{10}\sqrt{3}} = \dfrac{4}{\sqrt{30}}$

$\therefore$ रेखा MN तथा समतल के मध्य कोण $\sin^{-1}\dfrac{4}{\sqrt{30}}$ है।

E-6. The reflection of line $\dfrac{x-1}{3} = \dfrac{y-2}{4} = \dfrac{z-3}{5}$ about the plane $x - 2y + z - 6 = 0$ is

रेखा $\dfrac{x-1}{3} = \dfrac{y-2}{4} = \dfrac{z-3}{5}$ का समतल $x - 2y + z - 6 = 0$ के सापेक्ष प्रतिबिम्ब होगा।

Ans. $\dfrac{x-3}{3} = \dfrac{y+2}{4} = \dfrac{z-5}{5}$

Sol. $\therefore$ $3.1 - 2.4 + 5 \times 1 = 0$, line is parallel to the plane

$\therefore$ reflection of line will also have same direction ratios i.e. 3, 4, 5

Also mirror image of (1, 2, 3) will be on required line.

$$\frac{x-1}{1} = \frac{y-2}{-2} = \frac{z-3}{1} = -2\left(\frac{1-4+3-6}{1^2+1^2+(-2)^2}\right) \Rightarrow (x, y, z) = (3, -2, 5)$$

$\therefore$ equation of straight line $\dfrac{x-3}{3} = \dfrac{y+2}{4} = \dfrac{z-5}{5}$

Hindi $\therefore$ $3.1 - 2.4 + 5 \times 1 = 0$, रेखा समतल के समान्तर है।

$\therefore$ अतः रेखा के प्रतिबिम्ब के दिक् अनुपात वही अर्थात् 3, 4, 5 होगें।

पुनः (1, 2, 3) का दर्पण प्रतिबिम्ब अभीष्ठ रेखा पर होगा। $\dfrac{x-1}{1} = \dfrac{y-2}{-2} = \dfrac{z-3}{1} = -2\left(\dfrac{1-4+3-6}{1^2+1^2+(-2)^2}\right)$

$\Rightarrow (x, y, z) = (3, -2, 5)$

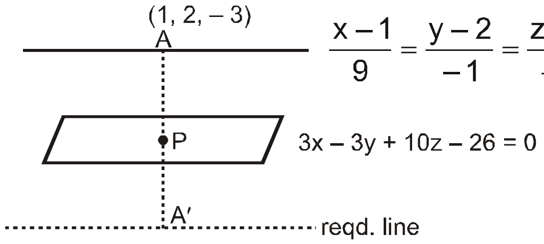
$\therefore$ सरल रेखा का समीकरण $\dfrac{x-3}{3} = \dfrac{y+2}{4} = \dfrac{z-5}{5}$

E-7. Find the equation of image of the line $\dfrac{x-1}{9} = \dfrac{y-2}{-1} = \dfrac{z+3}{-3}$ in the plane $3x - 3y + 10z = 26$.

रेखा $\dfrac{x-1}{9} = \dfrac{y-2}{-1} = \dfrac{z+3}{-3}$ का समतल $3x - 3y + 10z = 26$ में प्रतिबिम्ब का समीकरण ज्ञात कीजिए।**[16JM120012]**

Ans. $\dfrac{x-4}{9} = \dfrac{y+1}{-1} = \dfrac{z-7}{-3}$

Sol.



$\dfrac{x-1}{9} = \dfrac{y-2}{-1} = \dfrac{z+3}{-3}$

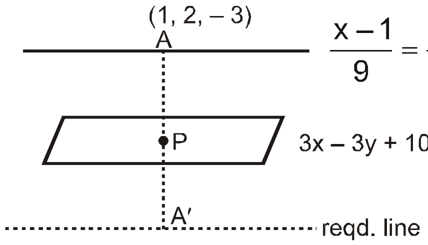
$\because$ A′ is mirror image of A w.r.t the given plane

$$\therefore \quad \frac{x-1}{3} = \frac{y-2}{-3} = \frac{z+3}{10} = -2\left(\frac{3-6-30-26}{9+9+100}\right)$$

$$\frac{x-1}{3} = \frac{y-2}{-3} = \frac{z+3}{10} = -2\left(-\frac{59}{118}\right) \quad \therefore \quad A'(4, -1, 7)$$

$\therefore$ Equation of required line $\dfrac{x-4}{9} = \dfrac{y+1}{-1} = \dfrac{z-7}{-3}$

Hindi.



$$\frac{x-1}{9}=\frac{y-2}{-1}=\frac{z+3}{-3}$$

$3x - 3y + 10z - 26 = 0$

$\because$ A′ दिये गये समतल के सापेक्ष A का प्रतिबिम्ब है।

$\therefore$ $\frac{x-1}{3}=\frac{y-2}{-3}=\frac{z+3}{10}=-2\left(\frac{3-6-30-26}{9+9+100}\right)$

$\frac{x-1}{3}=\frac{y-2}{-3}=\frac{z+3}{10}=-2\left(-\frac{59}{118}\right)$ $\therefore$ A′(4, – 1, 7)

$\therefore$ अभीष्ट रेखा का समीकरण $\frac{x-4}{9}=\frac{y+1}{-1}=\frac{z-7}{-3}$

E-8. Find the angle between the plane passing through points (1, 1, 1), (1, – 1, 1), (–7, –3, –5) & x–z plane.
बिन्दुओं (1, 1, 1), (1, – 1, 1), (–7, –3, –5) से गुजरने वाले समतल और x-z समतल के मध्य कोण ज्ञात कीजिए।
Ans. $\pi/2$

Sol. Equation of plane passing through (1, 1, 1) is $a(x - 1) + b(y - 1) + c(z - 1) = 0$ ... (1)

$\because$ it passes through (1, – 1, 1) and (– 7, – 3, – 5)

$\therefore$ $a.0 - 2.b + 0.c = 0$ $\Rightarrow$ $b = 0$ and $-8a - 4b - 6c = 0$

$4a + 2b + 3c = 0$ $\because$ $b = 0$

$\therefore$ $4a + 3c = 0$ $\Rightarrow$ $c = -\frac{4a}{3}$

$\therefore$ dr's of normal to the plane are 1, 0 – $\frac{4}{3}$ and dr's of the normal to the x-z plane are 0, 1, 0

$\therefore$ $\cos\theta = \left|\frac{0+0+0}{\sqrt{\Sigma a^2}\ \sqrt{\Sigma a_1^2}}\right| = 0$ $\therefore$ $\theta = \frac{\pi}{2}$

Hindi. (1, 1, 1) से गुजरने वाले समतल का समीकरण है $a(x - 1) + b(y - 1) + c(z - 1) = 0$ ... (1)

$\because$ यह (1, – 1, 1) और (– 7, – 3, – 5) से गुजरता है।

$\therefore$ $a.0 - 2.b + 0.c = 0$ $\Rightarrow$ $b = 0$ और $-8a - 4b - 6c = 0$ $\Rightarrow 4a + 2b + 3c = 0$

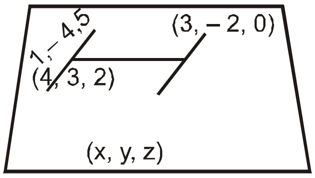
$\because$ $b = 0$ $\therefore$ $4a + 3c = 0$ $\Rightarrow$ $c = -\frac{4a}{3}$

$\therefore$ समतल के अभिलम्ब के दिक्अनुपात 1, 0 – $\frac{4}{3}$ है। और x-z समतल के अभिलम्ब के दिक्अनुपात 0, 1, 0 हैं।

$\therefore$ $\cos\theta = \left|\frac{0+0+0}{\sqrt{\Sigma a^2}\ \sqrt{\Sigma a_1^2}}\right| = 0$ $\therefore$ $\theta = \frac{\pi}{2}$

E-9. Find the equation of the plane containing parallel lines $(x - 4) = \frac{3-y}{4} = \frac{z-2}{5}$ and $(x - 3) = \lambda\,(y + 2) = \mu z$
उस समतल का समीकरण ज्ञात कीजिए जिसमें समान्तर रेखाएँ $(x - 4) = \frac{3-y}{4} = \frac{z-2}{5}$ और $(x - 3) = \lambda\,(y + 2) = \mu z$ निहित हैं।
Ans. $11x - y - 3z = 35$

Sol.



$\dfrac{x-4}{1} = \dfrac{y-3}{-4} = \dfrac{z-2}{5}$ .... (1)

$\dfrac{x-3}{1} = \dfrac{y+2}{1/\lambda} = \dfrac{z-0}{1/\mu}$ .... (2)

Equation of the plane is समतल का समीकरण है —

$\begin{vmatrix} x-3 & y+2 & z \\ 1 & 5 & 2 \\ 1 & -4 & 5 \end{vmatrix} = 0 \Rightarrow (x-3)(25+8)-(y+2)(5-2)+z(-4-5)=0$

$33x - 99 - 3y - 6 - 9z = 0 \Rightarrow 33x - 3y - 9z - 105 = 0 \Rightarrow 11x - y - 3z = 35$

E-10. Find the distance of the point (2, 3, 4) from the plane $3x + 2y + 2z + 5 = 0$, measured parallel to the line $\dfrac{x+3}{3} = \dfrac{y-2}{6} = \dfrac{z}{2}$ **[16JM120013]**

बिन्दु (2, 3, 4) की समतल $3x + 2y + 2z + 5 = 0$ से दूरी, रेखा $\dfrac{x+3}{3} = \dfrac{y-2}{6} = \dfrac{z}{2}$ के अनुदिश ज्ञात कीजिए।

Ans. 7

Sol. The given plane is $3x + 2y + 2z + 5 = 0$ ...(i)

Line through P(2, 3, 4) and parallel to the line $\dfrac{x+3}{3} = \dfrac{y-2}{6} = \dfrac{z}{2}$ is

$\dfrac{x-2}{3} = \dfrac{y-3}{6} = \dfrac{z-4}{2} = k$(say) ...(ii)

Any point on it is Q(3k + 2, 6k + 3, 2k + 4). Let it lie on (i)

$\therefore\ 3(3k + 2) + 2(6k + 3) + 2(2k + 4) + 5 = 0 \Rightarrow 25k + 25 = 0 \Rightarrow k = -1 \quad \therefore\ Q(-1, -3, 2)$

$\therefore$ The required distance = PQ $= \sqrt{(2+1)^2+(3+3)^2+(4-2)^2} = \sqrt{49} = 7$

Hindi दिया गया समतल है। $3x + 2y + 2z + 5 = 0$ ...(i)

रेखा जो P(2, 3, 4) से गुजरे व दी गई रेखा के समान्तर है।

$\dfrac{x+3}{3} = \dfrac{y-2}{6} = \dfrac{z}{2}$ is $\dfrac{x-2}{3} = \dfrac{y-3}{6} = \dfrac{z-4}{2} = k$ (say) ...(ii)

इस पर कोई बिन्दु Q(3k + 2, 6k + 3, 2k + 4)

Let it lie on (i)

$\therefore\ 3(3k + 2) + 2(6k + 3) + 2(2k + 4) + 5 = 0 \Rightarrow 25k + 25 = 0 \Rightarrow k = -1$

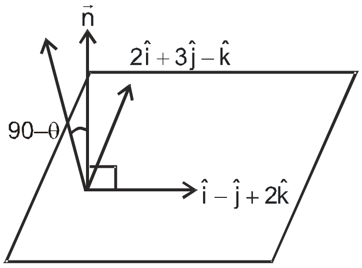
$\therefore$ Q(–1, –3, 2) $\therefore$ अभीष्ट दूरी = PQ $= \sqrt{(2+1)^2+(3+3)^2+(4-2)^2} = \sqrt{49} = 7$.

E-11. If the acute angle that the vector, $\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}$ makes with the plane of the two vectors $2\hat{i}+3\hat{j}-\hat{k}$ and $\hat{i}-\hat{j}+2\hat{k}$ is $\cot^{-1}\sqrt{2}$ then find the value of $\alpha(\beta+\gamma)-\beta\gamma$

सदिश $\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}$ द्वारा दो सदिशों $2\hat{i}+3\hat{j}-\hat{k}$ तथा $\hat{i}-\hat{j}+2\hat{k}$ के समतल के साथ बनाया गया न्यूनकोण $\cot^{-1}\sqrt{2}$ है, तो $\alpha(\beta+\gamma)-\beta\gamma$ ज्ञात कीजिए।

Ans. 0

Sol.



$\perp^r$ vector to plane $\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ 1 & -1 & 2 \end{vmatrix}$

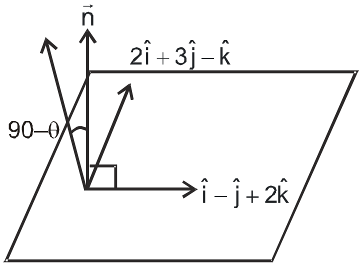
$\vec{n} = 5\hat{i} - 5\hat{j} - 5\hat{k}$

$\therefore \quad \cos(90-\theta) = \dfrac{(\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k})}{\sqrt{\alpha^2+\beta^2+\gamma^2}} \cdot \dfrac{(\hat{i}-\hat{j}-\hat{k})}{\sqrt{3}}$

$\sin\theta = \dfrac{\alpha-\beta-\gamma}{\sqrt{3}\;\sqrt{\alpha^2+\beta^2+\gamma^2}}$ $\qquad (\because \;\theta = \cot^{-1}\sqrt{2} \;\Rightarrow\; \cot\theta = \sqrt{2} \;\Rightarrow\; \sin\theta = \dfrac{1}{\sqrt{3}})$

$\Rightarrow \quad \dfrac{1}{\sqrt{3}} = \dfrac{\alpha-\beta-\gamma}{\sqrt{3}\;\sqrt{\alpha^2+\beta^2+\gamma^2}} \quad \Rightarrow \quad \alpha(\beta+\gamma) - \beta\gamma = 0$

Hindi



समतल के लम्बवत् सदिश $\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ 1 & -1 & 2 \end{vmatrix}$

$\vec{n} = 5\hat{i} - 5\hat{j} - 5\hat{k}$

$\therefore \quad \cos(90-\theta) = \dfrac{(\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k})}{\sqrt{\alpha^2+\beta^2+\gamma^2}} \cdot \dfrac{(\hat{i}-\hat{j}-\hat{k})}{\sqrt{3}}$

$\sin\theta = \dfrac{\alpha-\beta-\gamma}{\sqrt{3}\;\sqrt{\alpha^2+\beta^2+\gamma^2}}$ $\qquad (\because \;\theta = \cot^{-1}\sqrt{2} \;\Rightarrow\; \cot\theta = \sqrt{2} \;\Rightarrow\; \sin\theta = \dfrac{1}{\sqrt{3}})$

$\Rightarrow \quad \dfrac{1}{\sqrt{3}} = \dfrac{\alpha-\beta-\gamma}{\sqrt{3}\;\sqrt{\alpha^2+\beta^2+\gamma^2}} \quad \Rightarrow \quad \alpha(\beta+\gamma) - \beta\gamma = 0$

E-12. Find the equation of the plane passing through the points (3, 4, 1) and (0, 1, 0) and parallel to the line $\dfrac{x+3}{2} = \dfrac{y-3}{7} = \dfrac{z-2}{5}$. **[16JM120014]**

उस समतल का समीकरण ज्ञात कीजिए जो कि बिन्दुओं (3, 4, 1) तथा (0, 1, 0) से गुजरता है तथा रेखा $\dfrac{x+3}{2} = \dfrac{y-3}{7} = \dfrac{z-2}{5}$ के समान्तर है।

Ans. $8x - 13y + 15z + 13 = 0$

Sol. $\therefore$ Equation of plane through (3, 4, 1) is
$\therefore$ (3, 4, 1) से गुजरने वाला समतल है
$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \quad \Rightarrow \quad a(x - 3) + b(y - 4) + c(z - 1) = 0$ ...(i)
$\because$ plane (i) passes through (0, 1, 0) $\because$ समतल (i) बिन्दु (0, 1, 0) से गुजरता है
$\therefore -3a - 3b - c = 0 \quad \Rightarrow \quad 3a + 3b + c = 0$ ...(ii)
Also the plane (i) is parallel to the line $\frac{x+3}{2} = \frac{y-3}{7} = \frac{z-2}{5}$
तथा समतल (i) रेखा $\frac{x+3}{2} = \frac{y-3}{7} = \frac{z-2}{5}$ के समान्तर है।
$\therefore a_1 a_2 + b_1 b_2 + c_1 c_2 = 0 \quad \Rightarrow \quad 2a + 7b + 5c = 0$ ...(iii)
Eliminating a, b, c from (i), (ii) and (iii), we get
(i), (ii) तथा (iii) से a, b, c विलुप्त करने पर हम प्राप्त करते है

$$\begin{vmatrix} x-3 & y-4 & z-1 \\ 3 & 3 & 1 \\ 2 & 7 & 5 \end{vmatrix} = 0 \quad \Rightarrow \quad (x - 3)(15 - 7) - (y - 4)(15 - 2) + (z - 1)(21 - 6) = 0$$

$8(x - 3) - 13(y - 4) + 15(z - 1) = 0 \quad \Rightarrow \quad 8x - 24 - 13y + 52 + 15z - 15 = 0$

E-13. Find the vector equation of a line passing through the point with position vector $(2\hat{i} - 3\hat{j} - 5\hat{k})$ and perpendicular to the plane $\vec{r}.(6\hat{i} - 3\hat{j} + 5\hat{k}) + 2 = 0$.
Also, find the point of intersection of this line and the plane.
एक बिन्दु जिसका स्थिति सदिश $(2\hat{i} - 3\hat{j} - 5\hat{k})$ है, से गुजरने वाली तथा समतल $\vec{r}.(6\hat{i} - 3\hat{j} + 5\hat{k}) + 2 = 0$ के लम्बवत् सरल रेखा का सदिश समीकरण ज्ञात कीजिए।
साथ ही सरल रेखा तथा समतल का प्रतिच्छेदी बिन्दु भी ज्ञात कीजिये।

Ans. $\vec{r} = 2\hat{i} - 3\hat{j} - 5\hat{k} + t(6\hat{i} - 3\hat{j} + 5\hat{k})$, $\left(\frac{76}{35}, \frac{-108}{35}, \frac{-170}{35}\right)$

Sol. The required line is to pass through $\vec{a} = 2\hat{i} - 3\hat{j} - 5\hat{k}$ and is $\perp$ to the plane given by
$\vec{r}.(6\hat{i} - 3\hat{j} + 5\hat{k}) + 2 = 0$ ...(1)
Here the vector $\vec{b} = 6\hat{i} - 3\hat{j} + 5\hat{k}$ is normal to the plane (1)
$\Rightarrow$ The reqd. line is along the direction of this vector. Hence its equation is
$\vec{r} = \vec{a} + t\vec{b} \quad \Rightarrow \quad \vec{r} = 2\hat{i} - 3\hat{j} - 5\hat{k} + t(6\hat{i} - 3\hat{j} + 5\hat{k})$ ...(2)
Now line (2) meets the plane (1), when $\vec{r}.[2\hat{i} - 3\hat{j} + 5\hat{k} + t(6\hat{i} - 3\hat{j} + 5\hat{k})].(6\hat{i} - 3\hat{j} + 5\hat{k}) = -2$
i.e. when $6(2 + 6t) + 3(3 + 3t) + 5(5t - 5) = -2$ i.e., when $t = \frac{1}{35}$
Substituting this value of t in (2), we get $\vec{r} = 2\hat{i} - 3\hat{j} + 5\hat{k} + \frac{1}{35}(6\hat{i} - 3\hat{j} + 5\hat{k}) = \frac{1}{35}(76\hat{i} - 108\hat{j} - 170\hat{k})$
$\therefore$ The reqd. point of intersection is $\left(\frac{76}{35}, \frac{-108}{35}, \frac{-170}{35}\right)$

Hindi. अभीष्ट रेखा बिन्दु $\vec{a} = 2\hat{i} - 3\hat{j} - 5\hat{k}$ से गुजरती है तथा दिये गये समतल के लम्बवत् है
$\vec{r}.(6\hat{i} - 3\hat{j} + 5\hat{k}) + 2 = 0$ ...(1)
यहाँ सदिश $\vec{b} = 6\hat{i} - 3\hat{j} + 5\hat{k}$ समतल (1) का अभिलम्ब सदिश है।
$\Rightarrow$ अभीष्ट रेखा इस सदिश के अनुदिश है अतः इसका समीकरण है।
$\vec{r} = \vec{a} + t\vec{b} \quad \Rightarrow \quad \vec{r} = 2\hat{i} - 3\hat{j} - 5\hat{k} + t(6\hat{i} - 3\hat{j} + 5\hat{k})$ ...(2)
अब रेखा (2) समतल (1) को मिलती है यदि

$\vec{r}.[2\hat{i}-3\hat{j}+5\hat{k}+t(6\hat{i}-3\hat{j}+5\hat{k})].(6\hat{i}-3\hat{j}+5\hat{k}) = -2$ जब $6(2 + 6t) + 3(3 + 3t) + 5(5t - 5) = -2$

जब $t = \frac{1}{35}$; t का यह मान (2) में रखने पर हमें प्राप्त होता है।

$\vec{r} = 2\hat{i}-3\hat{j}+5\hat{k} + \frac{1}{35}(6\hat{i}-3\hat{j}+5\hat{k}) = \frac{1}{35}(76\hat{i}-108\hat{j}-170\hat{k})$ $\therefore$ अभीष्ट प्रतिच्छेदी बिन्दु है।

$\left(\frac{76}{35}, \frac{-108}{35}, \frac{-170}{35}\right)$

E-14. Find the equation of the plane passing through the point (1, 2, 1) and perpendicular to the line joining the points (1, 4, 2) and (2, 3, 5). Also find the coordinates of the foot of the perpendicular and the perpendicular distance of the point (4, 0, 3) from the above found plane. **[16JM120015]**

बिन्दु (1, 2, 1) से गुजरने वाले तथा बिन्दुओं (1, 4, 2) तथा (2, 3, 5) को मिलाने वाली रेखा के लम्बवत् समतल का समीकरण ज्ञात करो। इस समतल पर बिन्दु (4, 0, 3) से डाले गये लम्बपाद के निर्देशांक तथा लम्ब की लम्बाई भी ज्ञात कीजिए।

Ans. $x - y + 3z - 2 = 0$; (3, 1, 0) ; $\sqrt{11}$

Sol. Let P(1, 2, 1) Q(1, 4, 2) and R(2, 3, 5). Now d.r.s. of the line QR are $2 - 1, 3 - 4, 5 - 2 = 1, -1, 3$

The reqd. plane is to pass through 'P' and $\perp$ QR. Its equation is $1(x - 1) - 1 \cdot (y - 2) + 3(z - 1) = 0$

$\Rightarrow x - y + 3z - 2 = 0$ ...(i)

Let B(α, β, γ) be the foot of the $\perp$ from A(4, 0, 3) on (1).

$\therefore \alpha - \beta + 3\gamma - 2 = 0$ ...(ii)

and drs. of AB (Normal to the plane (i) are) $\alpha - 4, \beta - 0, \gamma - 3$

Hence $\frac{\alpha-4}{1} = \frac{\beta-0}{-1} = \frac{\gamma-3}{3} = k$ (say) $\Rightarrow$ $\alpha = k + 4, \beta = -k, \gamma = 3k + 3$ ...(iii)

From (ii) and (iii)

$(k + 4) + k + 3(3k + 3) - 2 = 0 \Rightarrow k = -1$

$\therefore$ The coordinates of the foot of $\perp$ from A on (1) is B(α, β, γ) = (3, 1, 0)

and $\perp$ = AB = $\sqrt{(4-3)^2+(0-1)^2+(3-0)^2} = \sqrt{11}$

Hindi. माना P(1, 2, 1) Q(1, 4, 2) तथा R(2, 3, 5) है अब रेखा QR के दिक्अनुपात $2 - 1, 3 - 4, 5 - 2 = 1, -1, 3$

अभीष्ट समतल बिन्दु 'P' से गुजरेगा तथा सरल रेखा QR के लम्बवत् होगा। इस समतल का समीकरण

$1(x - 1) - 1 \cdot (y - 2) + 3(z - 1) = 0$ $\Rightarrow x - y + 3z - 2 = 0$ ...(i)

माना बिन्दु A(4, 0, 3) से समतल (i) पर डाला गया लम्ब पाद B(α, β, γ) है। $\therefore$ $\alpha - \beta + 3\gamma - 2 = 0$ ...(ii)

(समतल (i) के अभिलम्ब) के दिक् अनुपात $\alpha - 4, \beta - 0, \gamma - 3$

अतः $\frac{\alpha-4}{1} = \frac{\beta-0}{-1} = \frac{\gamma-3}{3} = k$ (माना) $\Rightarrow \alpha = k + 4, \beta = -k, \gamma = 3k + 3$ ...(iii)

समीकरण (ii) तथा (iii) से

$(k + 4) + k + 3(3k + 3) - 2 = 0 \Rightarrow k = -1$

$\therefore$ A से समतल (i) पर डाले गये लम्ब के पाद के निर्देशांक B(α, β, γ) = (3, 1, 0)

तथा लम्ब की लम्बाई AB = $\sqrt{(4-3)^2+(0-1)^2+(3-0)^2} = \sqrt{11}$

E-15. Find the equation of the planes passing through points (1, 0, 0) and (0, 1, 0) and making an angle of 0.25 π radians with plane $x + y - 3 = 0$

उन समतलों के समीकरण ज्ञात कीजिए जो बिन्दुओं (1, 0, 0) और (0, 1, 0) से गुजरते हैं और समतल $x + y - 3 = 0$ के साथ 0.25 π रेडियन का कोण बनाते हैं।

Ans. $x+y \pm \sqrt{2}\ z = 1$

Sol. Equation of plane passing through (1,0,0) is : $a(x - 1) + b(y - 0) + c(z - 0) = 0$

$a(x - 1) + b(y) + c(z) = 0$ ....(1)

(1) also passes through (0,1,0)

$-a + b + c.0 = 0$ ....(2)

angle between plane (1) and $x + y - 3 = 0$ is

$\cos\frac{\pi}{4} = \left|\frac{a.1+b.1+c.0}{\sqrt{\Sigma a^2}\sqrt{1+1+0}}\right| = \frac{1}{\sqrt{2}} \Rightarrow \left|\frac{a+b}{\sqrt{a^2+b^2+c^2}\sqrt{2}}\right| = \frac{1}{\sqrt{2}}$

$(a + b)^2 = a^2 + b^2 + c^2 \Rightarrow 2ab = c^2$

from equation (2) $a = b \Rightarrow 2a^2 = c^2 \Rightarrow c = a\sqrt{2}$ or $c = -a\sqrt{2}$

So equation of planes are $x + y \pm \sqrt{2}z - 1 = 0$

Hindi (1,0,0) से गुजरने वाले समतल का समीकरण है $a(x-1) + b(y-0) + c(z-0) = 0$

$a(x-1) + b(y) + c(z) = 0$ ....(1)

समतल (1), (0,1,0) से भी गुजरता है

$-a + b + c.0 = 0$ ....(2)

समतल (1) तथा $x + y - 3 = 0$ के मध्य कोण है।

$\cos\frac{\pi}{4} = \left|\frac{a.1+b.1+c.0}{\sqrt{\Sigma a^2}\sqrt{1+1+0}}\right| = \frac{1}{\sqrt{2}} \Rightarrow \left|\frac{a+b}{\sqrt{a^2+b^2+c^2}\sqrt{2}}\right| = \frac{1}{\sqrt{2}}$

$(a + b)^2 = a^2 + b^2 + c^2$ $2ab = c^2$

समीकरण (2) से $a = b \Rightarrow 2a^2 = c^2 \Rightarrow c = a\sqrt{2}$ or $c = -a\sqrt{2}$

अतः समतलों के समीकरण $x + y \pm \sqrt{2}z - 1 = 0$ हैं।

E-16. Find the distance between the parallel planes $\vec{r}\,.(2\hat{i}-3\hat{j}+6\hat{k}) = 5$ and $\vec{r}\,.(6\hat{i}-9\hat{j}+18\hat{k}) + 20 = 0$.

समान्तर समतलों . $\vec{r}\,.(2\hat{i}-3\hat{j}+6\hat{k}) = 5$ और $\vec{r}\,.(6\hat{i}-9\hat{j}+18\hat{k}) + 20 = 0$ के बीच की दूरी ज्ञात कीजिए।

Ans. $\frac{5}{3}$ unit **[16JM120016]**

Sol. Planes are : $2x - 3y + 6z - 5 = 0 \Rightarrow 2x - 3y + 6z + \frac{20}{3} = 0$

Distance between planes is $= \left|\frac{-5-\frac{20}{3}}{\sqrt{(2)^2+(-3)^2+(6)^2}}\right| = \frac{5}{3}$

Hindi. समतल हैं : $2x - 3y + 6z - 5 = 0 \Rightarrow 2x - 3y + 6z + \frac{20}{3} = 0$

समतलों के बीच की दूरी $= \left|\frac{-5-\frac{20}{3}}{\sqrt{(2)^2+(-3)^2+(6)^2}}\right| = \frac{5}{3}$

E-17. If the planes $x - cy - bz = 0$, $cx - y + az = 0$ and $bx + ay - z = 0$ pass through a straight line, then find the value of $a^2 + b^2 + c^2 + 2abc$ is :

यदि समतल $x - cy - bz = 0$, $cx - y + az = 0$ तथा $bx + ay - z = 0$ एक सरल रेखा से गुजरते हैं, तो $a^2 + b^2 + c^2 + 2abc$ का मान ज्ञात कीजिए –

Ans. 1

Sol. $\begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0 \Rightarrow (1 - a^2) + c(-c - ab) - b(ac + b) = 0 \Rightarrow 1 - a^2 - c^2 - abc - abc - b^2 = 0$

$a^2 + b^2 + c^2 + 2abc = 1$

E-18. (i) If $\hat{n}$ is the unit vector normal to a plane and p be the length of the perpendicular from the origin to the plane, find the vector equation of the plane.

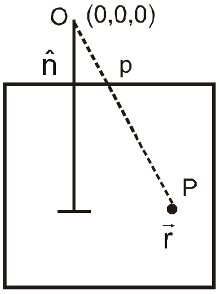
(ii) Find the equation of the plane which contains the origin and the line of intersection of the planes $\vec{r}\,.\vec{a} = p$ and $\vec{r}\,.\vec{b} = q$

(i) यदि $\hat{n}$ एक समतल के लम्बवत् इकाई सदिश है और मूलबिन्दु से समतल पर डाले गये लम्ब की लम्बाई p है, तो समतल की सदिश समीकरण ज्ञात कीजिए।

(ii) समतलों $\vec{r}.\vec{a} = p$ और $\vec{r}.\vec{b} = q$ की प्रतिच्छेदन रेखा और मूलबिन्दु को समाहित करने वाले समतल का समीकरण ज्ञात कीजिए।

Ans. (i) $\vec{r}.\hat{n} = \pm p$ (ii) $\vec{r}.(\vec{a}q - p\vec{b}) = 0$

Sol.



(i) Projection of OP on $\hat{n}$

OP का $\hat{n}$ पर प्रक्षेप

$\vec{r}.\hat{n} = p$

(ii) $\vec{r}.\vec{a} - p + \lambda(\vec{r}.\vec{b} - q) = 0$

$\vec{r} = \vec{0}$ $\therefore$ $-p - \lambda q = 0$ $\lambda = -\dfrac{p}{q}$

$\vec{r}.\vec{a} - p - \dfrac{p}{q}(\vec{r}.\ \vec{b} - q) = 0$ $\Rightarrow$ $\vec{r}.(\ \vec{a}q - p\vec{b}) = 0$

Exercise-2

Marked questions are recommended for Revision.
चिन्हित प्रश्न दोहराने योग्य प्रश्न है।

PART - I : ONLY ONE OPTION CORRECT TYPE
भाग-I : केवल एक सही विकल्प प्रकार (ONLY ONE OPTION CORRECT TYPE)

1. Let $\vec{a}, \vec{b}, \vec{c}$ be vectors of length 3, 4, 5 respectively. Let $\vec{a}$ be perpendicular to $\vec{b}+\vec{c}$, $\vec{b}$ to $\vec{c}+\vec{a}$ and $\vec{c}$ to $\vec{a}+\vec{b}$. Then $\left|\vec{a}+\vec{b}+\vec{c}\right|$ is equal to :

मानाकि सदिशों $\vec{a}, \vec{b}, \vec{c}$ की लम्बाई क्रमशः 3, 4, 5 है। माना $\vec{a}$ सदिश $\vec{b}+\vec{c}$ के, $\vec{b}$ सदिश $\vec{c}+\vec{a}$ के और $\vec{c}$ सदिश $\vec{a}+\vec{b}$ के लम्बवत है, तो $\left|\vec{a}+\vec{b}+\vec{c}\right|$ का मान है–

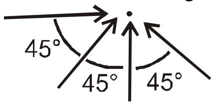
(A) $2\sqrt{5}$ (B) $2\sqrt{2}$ (C) $10\sqrt{5}$ (D*) $5\sqrt{2}$

Sol. $\vec{a}.(\vec{b}+\vec{c}) = 0$, $\vec{b}.(\vec{c}+\vec{a}) = 0$, $\vec{c}.(\vec{a}+\vec{b}) = 0$

$\Rightarrow \quad \vec{a}.\vec{b}+\vec{a}.\vec{c} = 0$, $\vec{b}.\vec{c}+\vec{b}.\vec{a} = 0$, $\vec{c}.\vec{a}+\vec{c}.\vec{b} = 0 \quad \Rightarrow \quad \vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a} = 0$

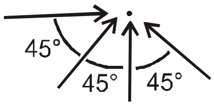
$|\vec{a}+\vec{b}+\vec{c}| = \sqrt{\vec{a}^2+\vec{b}^2+\vec{c}^2+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})} = \sqrt{9+16+25} = \sqrt{50}$

2. Four coplanar forces are applied at a point O. Each of them is equal to k and the angle between two consecutive forces equals 45º as shown in the figure. Then the resultant has the magnitude equal to :



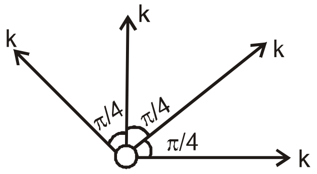
(A) $k\sqrt{2+2\sqrt{2}}$ (B) $k\sqrt{3+2\sqrt{2}}$ (C*) $k\sqrt{4+2\sqrt{2}}$ (D) $k\sqrt{4-2\sqrt{2}}$

एक बिन्दु O पर चार समतलीय बल लगाये गए है। इनमें से प्रत्येक k के बराबर है और दो क्रमागत बलों के मध्य कोण 45º है, जैसाकि चित्र में दर्शाया गया है, तो परिणामी बल का परिमाण है– **[16JM120030]**



(A) $k\sqrt{2+2\sqrt{2}}$ (B) $k\sqrt{3+2\sqrt{2}}$ (C*) $k\sqrt{4+2\sqrt{2}}$ (D) $k\sqrt{4-2\sqrt{2}}$

Sol.

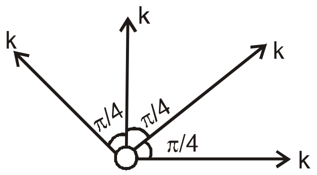


These forces can be written in terms of vector as $k\hat{i}$, $\frac{k}{\sqrt{2}}\hat{i}+\frac{k}{\sqrt{2}}\hat{j}$, $k\hat{j}$ and $-\frac{k}{\sqrt{2}}\hat{i}+\frac{k}{\sqrt{2}}\hat{j}$

Resultant = $k\hat{i}+(k+\sqrt{2}\ k)\hat{j}$

magnitude = $\sqrt{k^2+(k+\sqrt{2}\ k)^2} = k\sqrt{4+2\sqrt{2}}$

Hindi



बलों को सदिश रूप में निम्न प्रकार लिखा जा सकता है–

$k\hat{i},\ \ \frac{k}{\sqrt{2}}\hat{i}+\frac{k}{\sqrt{2}}\hat{j},\ k\hat{j}$ और $\quad -\frac{k}{\sqrt{2}}\hat{i}+\frac{k}{\sqrt{2}}\hat{j}$

परिणामी बल $= k\hat{i}+(k+\sqrt{2}\ \ k)\hat{j}$

परिमाण $\sqrt{k^2+(k+\sqrt{2}\ \ k)^2} = k\sqrt{4+2\sqrt{2}}$

3. Taken on side $\overrightarrow{AC}$ of a triangle ABC, a point M such that $\overrightarrow{AM} = \frac{1}{3}\overrightarrow{AC}$. A point N is taken on the side $\overrightarrow{CB}$ such that $\overrightarrow{BN} = \overrightarrow{CB}$, then for the point of intersection X of $\overrightarrow{AB}$ and $\overrightarrow{MN}$ which of the following holds good?

एक त्रिभुज ABC की भुजा $\overrightarrow{AC}$ पर एक बिन्दु M इस प्रकार लिया जाता है कि $\overrightarrow{AM} = \frac{1}{3}\overrightarrow{AC}$ तथा $\overrightarrow{CB}$ भुजा पर एक बिन्दु N इस प्रकार लिया जाता है कि $\overrightarrow{BN} = \overrightarrow{CB}$, तो $\overrightarrow{AB}$ और $\overrightarrow{MN}$ के प्रतिच्छेद बिन्दु X के लिए निम्न में से कौनसा कथन सत्य है ?

(A) $\overrightarrow{XB} = \frac{1}{3}\overrightarrow{AB}$ (B) $\overrightarrow{AX} = \frac{1}{3}\overrightarrow{AB}$ (C*) $\overrightarrow{XN} = \frac{3}{4}\overrightarrow{MN}$ (D) $\overrightarrow{XM} = 3\overrightarrow{XN}$

Sol. Position vector of M $\equiv \frac{\vec{c}}{3}$

Position vector of N $\equiv (-\vec{c}+2\vec{b})$

$\therefore$ equation of line BC is $\vec{r} = \vec{b}+\lambda(\vec{b}-\vec{c})$

$\therefore$ equation of line AB is is $\vec{r} = \vec{0}+\mu\vec{b}$

$\therefore$ equation of line MN is $\vec{r} = \frac{\vec{c}}{3}+t\left(\frac{4\vec{c}}{3}-2\right) \quad \Rightarrow \quad \mu = -2t,\ \ 0 = \frac{1}{3}+\frac{4}{3}t$

which gives $\mu = \frac{1}{2} \Rightarrow$ Position vector of X is $\frac{\vec{b}}{2}$.

Hindi M का स्थिति सदिश $\equiv \frac{\vec{c}}{3}$

N का स्थिति सदिश $\equiv (-\vec{c}+2\vec{b})$

$\therefore$ रेखा BC का समीकरण है $\vec{r} = \vec{b}+\lambda(\vec{b}-\vec{c})$

$\therefore$ रेखा AB का समीकरण है $\vec{r} = \vec{0}+\mu\vec{b}$

$\therefore$ रेखा MN का समीकरण है $\vec{r} = \frac{\vec{c}}{3}+t\left(\frac{4\vec{c}}{3}-2\right)$

$\Rightarrow \quad \mu = -2t,\ \ 0 = \frac{1}{3}+\frac{4}{3}t$ इससे प्राप्त होता है $\mu = \frac{1}{2} \Rightarrow$ X का स्थिति सदिश $\frac{\vec{b}}{2}$ है

4. If 3 non zero vectors $\vec{a},\ \vec{b},\ \vec{c}$ are such that $\vec{a}\times\vec{b} = 2(\vec{a}\times\vec{c})$, $|\vec{a}| = |\vec{c}| = 1$; $|\vec{b}| = 4$ the angle between $\vec{b}$ and $\vec{c}$ is $\cos^{-1}\frac{1}{4}$ then $\vec{b} = \ell\vec{c}+\mu\vec{a}$ where $|\ell|+|\mu|$ is - **[16JM120031]**

यदि तीन अशून्य सदिश $\vec{a},\ \vec{b},\ \vec{c}$ इस प्रकार है कि $\vec{a}\times\vec{b} = 2(\vec{a}\times\vec{c})$, $|\vec{a}| = |\vec{c}| = 1$; $|\vec{b}| = 4$ एवं $\vec{b}$ व $\vec{c}$ के मध्य कोण $\cos^{-1}\frac{1}{4}$ है, तो $\vec{b} = \ell\vec{c}+\mu\vec{a}$ जबकि $|\ell|+|\mu|$ है–

(A*) 6 (B) 5 (C) 4 (D) 0

Sol. $\vec{a}\times\vec{b} = 2(\vec{a}\times\vec{b}) \quad \Rightarrow \quad \vec{a}\times(\vec{b}-2\vec{c}) = 0 \quad \Rightarrow \quad \vec{b}-2\vec{c} = \alpha\vec{a}$

squaring (वर्ग करने पर) $b^2 + 4c^2 - 4\vec{b} \cdot \vec{c} = \alpha^2 a^2$ $\Rightarrow$ $16 + 4 - 4.4.1.\frac{1}{4} = \alpha^2$

$\Rightarrow$ $\alpha = \pm 4$ $\Rightarrow$ $\vec{b} = 2\vec{c} \pm 4\vec{a}$ $\Rightarrow$ $|\ell| + |\mu| = 6$

5. If θ is the angle between the vectors $\vec{p} = a\hat{i} + b\hat{j} + c\hat{k}$ and vector $\vec{q} = b\hat{i} + c\hat{j} + a\hat{k}$ then range of θ is

(A) $[0, \pi/3]$ (B) $[\pi/3, 2\pi/3]$ (C*) $[0, 2\pi/3]$ (D) $[0, 5\pi/6]$

यदि सदिश $\vec{p} = a\hat{i} + b\hat{j} + c\hat{k}$ और $\vec{q} = b\hat{i} + c\hat{j} + a\hat{k}$ के मध्य कोण θ है, तब θ के मानों का परिसर है–

(A) $[0, \pi/3]$ (B) $[\pi/3, 2\pi/3]$ (C*) $[0, 2\pi/3]$ (D) $[0, 5\pi/6]$

Sol. $\sum(a-b)^2 \geq 0 \Rightarrow \frac{ab+bc+ca}{a^2+b^2+c^2} \leq 1$ and $(a + b + c)^2 \geq 0 \Rightarrow \frac{ab+bc+ca}{a^2+b^2+c^2} \geq -\frac{1}{2}$

Now अब $\cos\theta = \frac{\vec{p}.\vec{q}}{|\vec{p}||\vec{q}|} = \frac{ab+bc+ca}{a^2+b^2+c^2}$ $\therefore$ $-\frac{1}{2} \leq \cos\theta \leq 1 \Rightarrow 0 \leq \theta \leq \frac{2\pi}{3}$

6. If the unit vectors $\vec{e}_1$ and $\vec{e}_2$ are inclined at an angle 2θ and $|\vec{e}_1 - \vec{e}_2| < 1$, then for $\theta \in [0, \pi]$, θ may lie in the interval :

यदि इकाई सदिशों $\vec{e}_1$ और $\vec{e}_2$ के बीच कोण 2θ है और $|\vec{e}_1 - \vec{e}_2| < 1$, तो $\theta \in [0, \pi]$ के लिए θ निम्न में से किस अन्तराल में हो सकता है –

(A*) $\left[0, \frac{\pi}{6}\right)$ (B) $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$ (C) $\left(\frac{5\pi}{6}, \pi\right]$ (D) $\left[\frac{\pi}{2}, \frac{5\pi}{6}\right]$

Sol. $|\vec{e}_1 - \vec{e}_2|^2 < 1$ $\Rightarrow$ $\vec{e}_1^{\,2} + \vec{e}_2^{\,2} - 2\vec{e}_1 . \vec{e}_2 < 1$ $\Rightarrow$ $1 + 1 - 2\cos(2\theta) < 1$

$\Rightarrow$ $2\cos2\theta > 1$ $\Rightarrow$ $\cos2\theta > \frac{1}{2}$ $\Rightarrow$ $2\theta \in \left[0, \frac{\pi}{3}\right) \Rightarrow \theta \in \left[0, \frac{\pi}{6}\right)$

7. A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube, then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$ is equal to

(A) 1/3 (B) 2/3 (C*) 4/3 (D) 5/3

एक रेखा घन के चार विकर्णों के साथ कोण $\alpha, \beta, \gamma, \delta$ कोण बनाती है तब $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$ बराबर है–

(A) 1/3 (B) 2/3 (C*) 4/3 (D) 5/3

Sol. Let direction cosines of line is ℓ, m, n which makes angle $\alpha, \beta, \gamma, \delta$ with diagonals of cube having direction cosines are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$, $\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ respectively.

So $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \left(\frac{\ell}{\sqrt{3}} + \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}}\right)^2 + \left(\frac{\ell}{\sqrt{3}} + \frac{m}{\sqrt{3}} - \frac{n}{\sqrt{3}}\right)^2 + \left(\frac{\ell}{\sqrt{3}} - \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}}\right)^2$

$+ \left(-\frac{\ell}{\sqrt{3}} + \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}}\right)^2 = \frac{4}{3}(\ell^2 + m^2 + n^2) = \frac{4}{3}$

8. Consider the lines $\frac{x}{2} = \frac{y}{3} = \frac{z}{5}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$, then the equation of the line which

(A) bisects the angle between the lines is $\frac{x}{3} = \frac{y}{3} = \frac{z}{8}$

(B) bisects the angle between the lines is $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$

(C*) passes through origin and is perpendicular to the given lines is x = y = – z

(D) none of these

दो रेखाऐं $\frac{x}{2} = \frac{y}{3} = \frac{z}{5}$ और $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ हैं, तब उस रेखा का समीकरण, जो

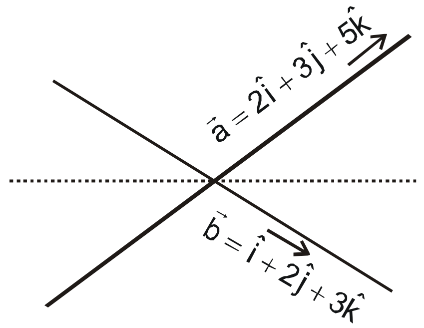
(A) दी गई रेखाओं के मध्य कोण को समद्विभाजित करती है, $\frac{x}{3} = \frac{y}{3} = \frac{z}{8}$ है।

(B) दी गई रेखाओं के मध्य कोण को समद्विभाजित करती है, $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ है।

(C) मूल बिन्दु से गुजरती है और दी गई रेखाओं x = y = – z के लम्बवत् है, है।

(D) इनमें से कोई नहीं

Sol.



$\frac{x}{2} = \frac{y}{3} = \frac{z}{5}$ **... (i)** $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ **... (ii)**

$\hat{a} + \hat{b} = \frac{2\hat{i}+3\hat{j}+5\hat{k}}{\sqrt{38}} + \frac{\hat{i}+2\hat{j}+3\hat{k}}{\sqrt{14}}$ $\Rightarrow$ (A) and (B) will be incorrect

Let the dr's of line $\perp$ to (1) and (2) be a, b, c

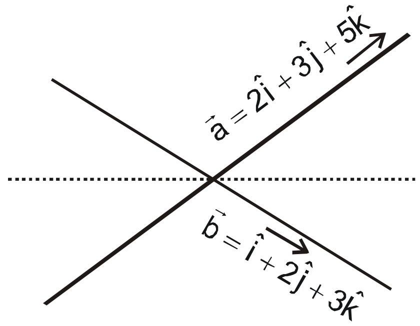
$\Rightarrow$ 2a + 3b + 5c = 0 **... (iii)** and a + 2b + 3c = 0 **... (iv)**

$\therefore$ $\frac{a}{9-10} = \frac{b}{5-6} = \frac{c}{4-3}$ $\Rightarrow$ $\frac{a}{-1} = \frac{b}{-1} = \frac{c}{1}$ $\Rightarrow$ $\frac{a}{1} = \frac{b}{1} = \frac{c}{-1}$

$\therefore$ equation of line passing through (0, 0, 0) and is $\perp$r to the lines (1) and (2) is

$\frac{x}{1} = \frac{y}{1} = \frac{z}{-1}$ **Ans.**

Hindi.



$\frac{x}{2} = \frac{y}{3} = \frac{z}{5}$ **... (i)** $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ **... (ii)**

$\hat{a} + \hat{b} = \frac{2\hat{i}+3\hat{j}+5\hat{k}}{\sqrt{38}} + \frac{\hat{i}+2\hat{j}+3\hat{k}}{\sqrt{14}}$ $\Rightarrow$ (A) और (B) सही नहीं हैं।

मानाकि (1) और (2) के लम्बवत् रेखा के दिक्अनुपात a, b, c हैं।

$\Rightarrow$ 2a + 3b + 5c = 0 **... (iii)** और a + 2b + 3c = 0 **... (iv)**

$\therefore$ $\frac{a}{9-10} = \frac{b}{5-6} = \frac{c}{4-3}$ $\Rightarrow$ $\frac{a}{-1} = \frac{b}{-1} = \frac{c}{1}$ $\Rightarrow$ $\frac{a}{1} = \frac{b}{1} = \frac{c}{-1}$

$\therefore$ (0, 0, 0) से गुजरने वाली रेखा का समीकरण जोकि रेखाओं (1) और (2) के लम्बवत् है।

$\frac{x}{1} = \frac{y}{1} = \frac{z}{-1}$ **Ans.**

9. Given $\vec{a} = x\hat{i} + y\hat{j} + 2\hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$, $\vec{c} = \hat{i} + 2\hat{j}$; $(\vec{a} \wedge \vec{b}) = \frac{\pi}{2}$, $\vec{a}.\vec{c} = 4$, then

दिया गया है कि $\vec{a} = x\hat{i} + y\hat{j} + 2\hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$, $\vec{c} = \hat{i} + 2\hat{j}$; $(\vec{a} \wedge \vec{b}) = \frac{\pi}{2}$, $\vec{a}.\vec{c} = 4$, तो

(A) $[\vec{a}\ \vec{b}\ \vec{c}]^2 = |\vec{a}|$ (B) $[\vec{a}\ \vec{b}\ \vec{c}] = |\vec{a}|$ (C) $[\vec{a}\ \vec{b}\ \vec{c}] = 0$ (D*) $[\vec{a}\ \vec{b}\ \vec{c}] = |\vec{a}|^2$

Sol. $\vec{a}.\vec{b} = 0 \Rightarrow x - y + 2 = 0$ ... (1)

$\vec{a}.\vec{c} = 4 \Rightarrow x + 2y = 4$ ... (2)

$\Rightarrow x = 0,\ y = 2$ Hence (अत:) $\vec{a} = 2\hat{j} + 2\hat{k} \Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 0 & 2 & 2 \\ 1 & -1 & 1 \\ 1 & 2 & 0 \end{vmatrix} = 8 = |\vec{a}|^2$

10. The vectors $\hat{i}+2\hat{j}+3\hat{k}$, $2\hat{i}-\hat{j}+\hat{k}$ and $3\hat{i}+\hat{j}+4\hat{k}$ are so placed that the end point of one vector is the starting point of the next vector. Then the vectors are : **[16JM120032]**

(A) Not coplaner (B*) Coplaner but cannot form a triangle

(C) Coplaner but can form a triangle (D) Coplaner & can form a right angled triangle

सदिश $\hat{i}+2\hat{j}+3\hat{k}$, $2\hat{i}-\hat{j}+\hat{k}$ और $3\hat{i}+\hat{j}+4\hat{k}$ इस प्रकार है कि एक सदिश का अन्तिम बिन्दु दूसरे सदिश का प्रारम्भिक बिन्दु हैं, तो सदिश हैं –

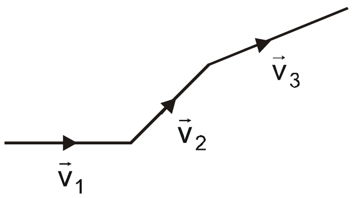
(A) असमतलीय (B) समतलीय पर त्रिभुज नहीं बना सकते है

(C) समतलीय लेकिन एक त्रिभुज बना सकते हैं। (D) समतलीय और एक समकोण त्रिभुज बना सकते हैं।

Sol. Clearly triangle is not possible as $v_1 + v_2 + v_3 \neq 0$

Since $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$

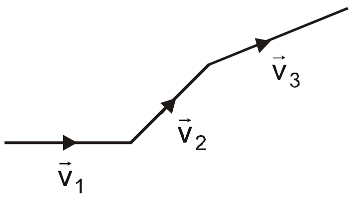
Hence $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3$ are coplaner



Hindi स्पष्टत: तीनों से त्रिभुज बनना सम्भव नहीं है जैसा कि $v_1 + v_2 + v_3 \neq 0$

$\because\ \vec{v}_3 = \vec{v}_1 + \vec{v}_2$

अत: $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3$ समतलीय है।



11. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be non-coplanar unit vectors equally inclined to one another at an acute angle θ. Then $\left|[\vec{a}\ \vec{b}\ \vec{c}]\right|$ in terms of θ is equal to: **[16JM120033]**

(A) $(1 + \cos\theta)\sqrt{\cos 2\theta}$ (B) $(1 + \cos\theta)\sqrt{1 - 2\cos 2\theta}$

(C*) $(1 - \cos\theta)\sqrt{1 + 2\cos\theta}$ (D) $(1 - \sin\theta)\sqrt{1 + 2\cos\theta}$

माना $\vec{a}, \vec{b}$ और $\vec{c}$ असमतलीय इकाई सदिश हैं और प्रत्येक एक दूसरे के साथ समान न्यून कोण θ बनाते है, तो $|[\vec{a}\ \vec{b}\ \vec{c}]|$ का मान θ के पदों में है–

(A) $(1 + \cos\theta)\sqrt{\cos 2\theta}$ (B) $(1 + \cos\theta)\sqrt{1-2\cos 2\theta}$

(C*) $(1 - \cos\theta)\sqrt{1+2\cos\theta}$ (D) $(1 - \sin\theta)\sqrt{1+2\cos\theta}$

Sol. $[\vec{a}\ \vec{b}\ \vec{c}]^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix} = \begin{vmatrix} 1 & \cos\theta & \cos\theta \\ \cos\theta & 1 & \cos\theta \\ \cos\theta & \cos\theta & 1 \end{vmatrix}$

12. Consider a tetrahedron with faces f_1, f_2, f_3, f_4. Let $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ be the vectors whose magnitudes are respectively equal to the areas of f_1, f_2, f_3, f_4 and whose directions are perpendicular to these faces in the outward direction. Then,

(A*) $\vec{a}_1 + \vec{a}_2 + \vec{a}_3 + \vec{a}_4 = \vec{0}$ (B) $\vec{a}_1 + \vec{a}_3 = \vec{a}_2 + \vec{a}_4$

(C) $\vec{a}_1 + \vec{a}_2 = \vec{a}_3 + \vec{a}_4$ (D) $\vec{a}_1 + \vec{a}_2 - \vec{a}_3 + \vec{a}_4 = \vec{0}$

एक चतुष्फलक के पृष्ठ f_1, f_2, f_3, f_4 हैं। मानाकि चार सदिश $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$ हैं जिनके परिमाण क्रमशः f_1, f_2, f_3, f_4 के क्षेत्रफलों के बराबर हैं और दिशा इन पृष्ठों के लम्बवत् बाहर की ओर है। तो,

(A*) $\vec{a}_1 + \vec{a}_2 + \vec{a}_3 + \vec{a}_4 = \vec{0}$ (B) $\vec{a}_1 + \vec{a}_3 = \vec{a}_2 + \vec{a}_4$

(C) $\vec{a}_1 + \vec{a}_2 = \vec{a}_3 + \vec{a}_4$ (D) $\vec{a}_1 + \vec{a}_2 - \vec{a}_3 + \vec{a}_4 = \vec{0}$

Sol. $\vec{a}_1 = \overrightarrow{AC} \times \overrightarrow{AB} = (\vec{c} - \vec{a}) \times (\vec{b} - \vec{a})$

$\vec{a}_2 = \overrightarrow{DB} \times \overrightarrow{DC} = \vec{b} \times \vec{c}$

$\vec{a}_3 = \overrightarrow{DC} \times \overrightarrow{DA} = \vec{c} \times \vec{a}$

$\vec{a}_4 = \overrightarrow{DA} \times \overrightarrow{DB} = \vec{a} \times \vec{b}$

13. Let $\vec{r}$ be a vector perpendicular to $\vec{a} + \vec{b} + \vec{c}$, where $[\vec{a}\ \vec{b}\ \vec{c}] = 2$. If **[16JM120034]**

$\vec{r} = \ell(\vec{b} \times \vec{c}) + m(\vec{c} \times \vec{a}) + n(\vec{a} \times \vec{b})$, then $(\ell + m + n)$ is equal to

(A) 2 (B) 1 (C*) 0 (D) –1

मानाकि $\vec{r}$ के $\vec{a} + \vec{b} + \vec{c}$ लम्बवत् एक सदिश है, जहाँ $[\vec{a}\ \vec{b}\ \vec{c}] = 2$. यदि

$\vec{r} = \ell(\vec{b} \times \vec{c}) + m(\vec{c} \times \vec{a}) + n(\vec{a} \times \vec{b})$ हो, तो $(\ell + m + n)$ का मान है–

(A) 2 (B) 1 (C) 0 (D) –1

Sol. $\vec{r}.(\vec{a}+\vec{b}+\vec{c}) = 0$

$\Rightarrow \quad \ell[\vec{a}\ \vec{b}\ \vec{c}] + m[\vec{a}\ \vec{b}\ \vec{c}] + n[\vec{a}\ \vec{b}\ \vec{c}] = 0$ or (या) $(\ell + m + n)\ [\vec{a}\ \vec{b}\ \vec{c}] = 0$

or (या) $\ell + m + n = 0$

14. If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar non-zero vectors and $\vec{r}$ is any vector in space, then $(\vec{a}\times\vec{b}) \times (\vec{r}\times\vec{c}) + (\vec{b}\times\vec{c}) \times (\vec{r}\times\vec{a}) + (\vec{c}\times\vec{a}) \times (\vec{r}\times\vec{b})$ is equal to

(A*) $2[\vec{a}\ \vec{b}\ \vec{c}]\vec{r}$ (B) $3[\vec{a}\ \vec{b}\ \vec{c}]\vec{r}$ (C) $[\vec{a}\ \vec{b}\ \vec{c}]\vec{r}$ (D) $4[\vec{a}\ \vec{b}\ \vec{c}]\vec{r}$

यदि $\vec{a}, \vec{b}, \vec{c}$ तीन असमतलीय, अशून्य सदिश हैं और समष्टि में कोई सदिश $\vec{r}$ है, तो $(\vec{a}\times\vec{b}) \times (\vec{r}\times\vec{c}) + (\vec{b}\times\vec{c}) \times (\vec{r}\times\vec{a}) + (\vec{c}\times\vec{a}) \times (\vec{r}\times\vec{b})$ का मान है–

(A*) $2[\vec{a}\ \vec{b}\ \vec{c}]\vec{r}$ (B) $3[\vec{a}\ \vec{b}\ \vec{c}]\vec{r}$ (C) $[\vec{a}\ \vec{b}\ \vec{c}]\vec{r}$ (D) $4[\vec{a}\ \vec{b}\ \vec{c}]\vec{r}$

Sol. $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$

Consider $(\vec{a}\times\vec{b}) \times (\vec{r}\times\vec{c}) = [\vec{a}\ \vec{b}\ \vec{c}]\ \vec{r} - [\vec{a}\ \vec{b}\ \vec{r}]\vec{c}$

But $[\vec{a}\ \vec{b}\ \vec{r}] = (\vec{a}\times\vec{b}).\vec{r} = z\ [\vec{a}\ \vec{b}\ \vec{c}]$ above expression $= [\vec{a}\ \vec{b}\ \vec{c}]\ (\vec{r} - z\vec{c})$

Hence sum $= [\vec{a}\ \vec{b}\ \vec{c}]\ (3\vec{r} - x\vec{a} - y\vec{b} - z\vec{c}) = [\vec{a}\ \vec{b}\ \vec{c}]\ 2\vec{r}$

Hindi. $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$

$\because \quad (\vec{a}\times\vec{b}) \times (\vec{r}\times\vec{c}) = [\vec{a}\ \ \vec{b}\ \ \vec{c}]\ \vec{r} - [\vec{a}\ \ \vec{b}\ \vec{r}]\vec{c}$

लेकिन $[\vec{a}\ \ \vec{b}\ \ \vec{r}] = (\vec{a}\times\vec{b}) . \vec{r} = z\ [\vec{a}\ \ \vec{b}\ \ \vec{c}]$ उपरोक्त व्यंजक = $[\vec{a}\ \ \vec{b}\ \ \vec{c}]\ (\vec{r} - z\vec{c})$

अतः योग = $[\vec{a}\ \ \vec{b}\ \ \vec{c}]\ (3\vec{r} - x\vec{a} - y\vec{b} - z\vec{c}) = [\vec{a}\ \ \vec{b}\ \ \vec{c}]\ 2\vec{r}$

15. If $\vec{b}$ and $\vec{c}$ are two non-collinear vectors such that $\vec{a} \,||\, (\vec{b} \times \vec{c})$, then $(\vec{a} \times \vec{b}) . (\vec{a} \times \vec{c})$ is equal to

(A*) $\vec{a}^2(\vec{b}.\vec{c})$ (B) $\vec{b}^2(\vec{a}.\vec{c})$ (C) $\vec{c}^2(\vec{a}.\vec{b})$ (D) $-\vec{a}^2(\vec{b}.\vec{c})$

यदि $\vec{b}$ और $\vec{c}$ दो असंरेखीय सदिश इस प्रकार है कि $\vec{a} \,||\, (\vec{b} \times \vec{c})$, तो $(\vec{a} \times \vec{b}) . (\vec{a} \times \vec{c})$ का मान है–

(A*) $\vec{a}^2(\vec{b}.\vec{c})$ (B) $\vec{b}^2(\vec{a}.\vec{c})$ (C) $\vec{c}^2(\vec{a}.\vec{b})$ (D) $-\vec{a}^2(\vec{b}.\vec{c})$

Sol. $\vec{a} \,||\, (\vec{b}\times\vec{c}) \Rightarrow \vec{a} = \lambda(\vec{b}\times\vec{c})$

also और $(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{c}) = \begin{vmatrix} \vec{a}\,.\,\vec{a} & \vec{a}\,.\,\vec{c} \\ \vec{b}\,.\,\vec{a} & \vec{b}\,.\,\vec{c} \end{vmatrix} = \begin{vmatrix} \vec{a}\,.\,\vec{a} & 0 \\ 0 & \vec{b}\,.\,\vec{c} \end{vmatrix} = |\vec{a}|^2\,(\vec{b}.\vec{c})$

16. Let $2\vec{c}+\vec{b} = 4\vec{a}+3\vec{d}$. If $\left[\vec{d}\ \vec{c}\ \vec{a}\right]$ and $\left[\vec{a}\ \vec{b}\ \vec{d}\right]$ are natural numbers with H.C.F. equal to 1 then how many statement are true among below six statement.

माना $2\vec{c}+\vec{b} = 4\vec{a}+3\vec{d}$ यदि $\left[\vec{d}\ \vec{c}\ \vec{a}\right]$ और $\left[\vec{a}\ \vec{b}\ \vec{d}\right]$ प्राकृत संख्याऐं है जिनका म.स.प. 1 है तब नीचे दिए छः कथनों में से कितने कथन सही है।

(i) $\left[\vec{a}\ \vec{b}\ \vec{d}\right] = 2$ (ii) $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 3$

(iii) $\left[\vec{d}\ \vec{c}\ \vec{b}\right] = 4$ (iv) $\left[\vec{d}\ \vec{c}\ \vec{a}\right] = 1$

(v) $\left(\vec{a}\times\vec{b}\right) \times \left(\vec{c}\times\vec{d}\right) = 2\vec{c} - 3\vec{d}$ (vi) $\left(\vec{a}\times\vec{b}\right) \times \left(\vec{c}\times\vec{d}\right) = 4\vec{a}-\vec{b}$

(A) 2 (B) 4 (C*) 6 (D) 0

Sol. $\left(2\vec{c}+\vec{b}\right).\left(\vec{a}\times\vec{d}\right) = \left(4\vec{a}+3\vec{d}\right).\left(\vec{a}\times\vec{d}\right)$

$\Rightarrow \quad 2\left[\vec{c}\ \vec{a}\ \vec{d}\right] + \left[\vec{b}\ \vec{a}\ \vec{d}\right] = 0 \qquad \Rightarrow \left[\vec{a}\ \vec{b}\ \vec{d}\right] = 2\left[\vec{d}\ \vec{c}\ \vec{a}\right]$ .....(i)

Now अब $\left(2\vec{c}+\vec{b}\right).\left(\vec{c}\times\vec{d}\right) = \left(4\vec{a}+3\vec{d}\right).\left(\vec{c}\times\vec{d}\right) \qquad \Rightarrow \left[\vec{d}\ \vec{c}\ \vec{b}\right] = 4\left[\vec{d}\ \vec{c}\ \vec{a}\right]$ .....(ii)

Now अब $\left(2\vec{c}+\vec{b}\right).\left(\vec{a}\times\vec{c}\right) = \left(4\vec{a}+3\vec{d}\right).\left(\vec{a}\times\vec{c}\right) \qquad \Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 3\left[\vec{d}\ \vec{c}\ \vec{a}\right]$ .....(iii)

From (i) (ii) and (iii) we get $\left[\vec{d}\ \vec{c}\ \vec{a}\right] = 1$, $\left[\vec{a}\ \vec{b}\ \vec{d}\right] = 2$, $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 3$, $\left[\vec{d}\ \vec{c}\ \vec{b}\right] = 4$

(i),(ii) और (iii) से $\left[\vec{d}\ \vec{c}\ \vec{a}\right] = 1$, $\left[\vec{a}\ \vec{b}\ \vec{d}\right] = 2$, $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 3$, $\left[\vec{d}\ \vec{c}\ \vec{b}\right] = 4$

{ $\because$ H.C.F. of $\left[\vec{a}\ \vec{b}\ \vec{d}\right]$ and और $\left[\vec{d}\ \vec{c}\ \vec{a}\right] = 1$}

Now अब $\left(\vec{a}\times\vec{b}\right) \times \left(\vec{c}\times\vec{d}\right) = \left[\vec{a}.\vec{b}.\vec{d}\right]\vec{c} - \left[\vec{a}.\vec{b}.\vec{c}\right]\vec{d} = 2\vec{c} - 3\vec{d} = 4\vec{a} - \vec{b}$

17. Let $\vec{u}$ and $\vec{v}$ are unit vectors and $\vec{w}$ is a vector such that $(\vec{u}\times\vec{v})+\vec{u} = \vec{w}$ and $\vec{w}\times\vec{u} = \vec{v}$ then the value of $\left[\vec{u}\ \ \vec{v}\ \ \vec{w}\right]$ is equal to

(A*) 1 (B) 2 (C) 0 (D) –1

माना $\vec{u}$ और $\vec{v}$ इकाई सदिश है तथा $\vec{w}$ एक सदिश इस प्रकार है कि $(\vec{u}\times\vec{v})+\vec{u} = \vec{w}$ तथा $\vec{w}\times\vec{u} = \vec{v}$, $\left[\vec{u}\ \ \vec{v}\ \ \vec{w}\right]$ मान बराबर है–

(A*) 1 (B) 2 (C) 0 (D) –1

Sol. Given दिया गया है $(\vec{u}\times\vec{v})+\vec{u} = \vec{w}$ and तथा $\vec{w}\times\vec{u} = \vec{v}$

$\Rightarrow \quad (\vec{u}\times\vec{v})+\vec{u} \times \vec{u} = \vec{w}\times\vec{u} \quad \Rightarrow \quad \left(\vec{u}\times\vec{v}\right) \times \vec{u}+\vec{u}\times\vec{u} = \vec{v}$ (as $\vec{w}\times\vec{u} = \vec{v}$)

$\Rightarrow \quad (\vec{u} \cdot \vec{u})\,\vec{v} - (v \cdot \vec{u})\,\vec{u} + \vec{u}\times\vec{u} = \vec{v}$

(using $\vec{u}\cdot\vec{u} = 1$ and और $\vec{u}\times\vec{u} = 0$, since unit vector से इकाई सदिश है)

$\Rightarrow \quad \vec{v} - (\vec{v} \cdot \vec{u})\,\vec{u} = \vec{v} \qquad \Rightarrow \quad (\vec{u} \cdot \vec{v})\,\vec{u} = \vec{0}$

$\Rightarrow \quad \vec{u} \cdot \vec{v} = 0$ (as $\vec{u} \neq 0$) ....(i)

Nowअब $\vec{u} \cdot (\vec{v}\times\vec{w})$

$= \vec{u} \cdot (\vec{v}\times((\vec{u}\times\vec{v})+\vec{u}))$ (given दिया है $\vec{w} = (\vec{u}\times\vec{v}) + u$)

$= \vec{u} \cdot (\vec{v}\times(\vec{u}\times\vec{v})+\vec{v}\times\vec{u}) = \vec{u} \cdot ((\vec{v} \cdot \vec{v})\,\vec{u} - (\vec{v} \cdot \vec{u})\,\vec{v} + \vec{v}\times\vec{u})$

$= \vec{u} \cdot (|\vec{v}|^2 u - 0 + \vec{v}\times\vec{u})$ (as $\vec{u} \cdot \vec{v} = 0$ from समीकरण (i) से)

$= |\vec{v}|^2 (\vec{u} \cdot \vec{u}) - \vec{u} \cdot (\vec{v}\times\vec{u})$

$= |\vec{v}|^2 |\vec{u}|^2 - 0$ (as $[\vec{u}\ \vec{v}\ \vec{u}] = 0$)

$= 1$ (as चूंकि $|\vec{u}| = |\vec{v}| = 1$)

$\therefore \quad [\vec{u}\ \vec{v}\ \vec{w}] = 1$

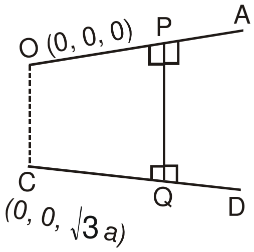
18 . Find the shortest distance between any two opposite edges of a tetrahedron formed by the planes $y + z = 0, x + z = 0, x + y = 0, x + y + z = \sqrt{3}\,a$.

(A) a (B) 2a (C) $a/\sqrt{2}$ (D*) $\sqrt{2}$ a.

समतलों $y + z = 0, x + z = 0, x + y = 0, x + y + z = \sqrt{3}\,a$ से बने चतुष्फलक के दो विपरीत किनारों के मध्य लघुत्तम दुरी है।

(A) a (B) 2a (C) $a/\sqrt{2}$ (D*) $\sqrt{2}$ a.

Sol.



Let the equation to one of the pair of opposite edges OA and BC be

$y + z = 0, x + z = 0$ .....(1) and $x + y = 0, x + y + z = \sqrt{3}\,a$ .....(2)

equation (1) and (2) can be expressed in symmetrical form as

$$\frac{x-0}{1} = \frac{y-0}{1} = \frac{z-0}{-1} \quad ..... (3) \quad \text{and,} \quad \frac{x-0}{1} = \frac{y-0}{-1} = \frac{z-\sqrt{3}\,a}{0} \quad ....(4)$$

d. r. of OA and BC are respectively (1, 1, – 1) and (1, – 1, 0).

Let PQ be the shortest distance between OA and BC having direction cosines (ℓ, m, n)

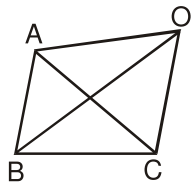
$\therefore$ PQ is perpendicular to both OA and BC.

$\therefore \quad \ell + m - n = 0$ and $\ell - m = 0$

Solving (5) and (6), we get, $\frac{\ell}{1} = \frac{m}{1} = \frac{n}{2} = k$ (say)

also, $\ell^2 + m^2 + n^2 = 1$

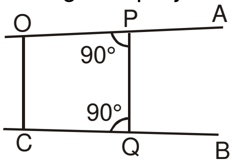
$\therefore \quad k^2 + k^2 + 4k^2 = 1 \Rightarrow k = \pm\frac{1}{\sqrt{6}}$



$\therefore \quad \ell = \pm\frac{1}{\sqrt{6}}, m = \pm\frac{1}{\sqrt{6}}, n = \pm\frac{2}{\sqrt{6}}$

Shortest distance between OA and BC

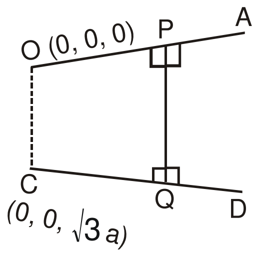
i.e. PQ = The length of projection of OC on PQ



$= |(x_2 - x_1)\,\ell + (y_2 - y_1)\,m + (z_2 - z_1)\,n|$

$= \left| 0 \;.\; \dfrac{1}{\sqrt{6}} + 0 \;.\; \dfrac{1}{\sqrt{6}} + \sqrt{3}\,a \;.\; \dfrac{2}{\sqrt{6}} \right| = \sqrt{2}\,a.$

Sol.



माना OA तथा BC विपरीत किनारों के युग्म मे से एक समीकरण है।

$y + z = 0,\ x + z = 0$ .....(1) तथा $x + y = 0,\ x + y + z = \sqrt{3}\,a$ .....(2)

समीकरण (1) तथा (2) को सममित रूप में लिखने पर

$\dfrac{x-0}{1} = \dfrac{y-0}{1} = \dfrac{z-0}{-1}$ ..... (3) तथा, $\dfrac{x-0}{1} = \dfrac{y-0}{-1} = \dfrac{z-\sqrt{3}\,a}{0}$ ....(4)

OA तथा BC के दिक अनुपात (1, 1, – 1) और (1, – 1, 0) है।

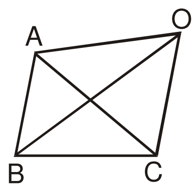
माना PQ, OA तथा BC के मध्य लघुत्तम दुरी है तथा दिककोज्याएं (ℓ, m, n) है। PQ, OA तथा BC दोनों के लम्बवत् है।

$\therefore$ $\ell + m - n = 0$ तथा $\ell - m = 0$

समीकरण (5) तथा (6) से, $\dfrac{\ell}{1} = \dfrac{m}{1} = \dfrac{n}{2} = k$ (say)

तथा, $\ell^2 + m^2 + n^2 = 1$

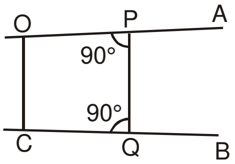
$\therefore$ $k^2 + k^2 + 4k^2 = 1 \Rightarrow k = \pm \dfrac{1}{\sqrt{6}}$



$\therefore$ $\ell = \pm \dfrac{1}{\sqrt{6}},\ m = \pm \dfrac{1}{\sqrt{6}},\ n = \pm \dfrac{2}{\sqrt{6}}$

OA तथा BC के मध्य लघुत्तम दुरी

अर्थात PQ = OC का PQ पर प्रक्षेप की लम्बाई



$= |(x_2 - x_1)\,\ell + (y_2 - y_1)\,m + (z_2 - z_1)\,n|$

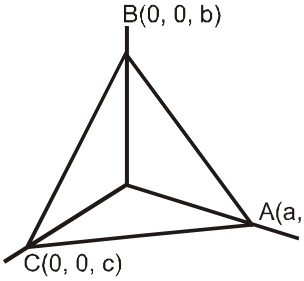
$$= \begin{vmatrix} 0 & . & \frac{1}{\sqrt{6}}+0 & . & \frac{1}{\sqrt{6}}+\sqrt{3}\ a & . & \frac{2}{\sqrt{6}} \end{vmatrix} = \sqrt{2}\ a.$$

19. A plane meets the coordinate axes in A, B, C and (α, β, γ) is the centroid of the triangle ABC, then the equation of the plane is **[16JM120036]**

(A*) $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=3$ (B) $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1$ (C) $\frac{3x}{\alpha}+\frac{3y}{\beta}+\frac{3z}{\gamma}=1$ (D) $\alpha x + \beta y + \gamma z = 1$

एक समतल निर्देशी अक्षों को A, B, C पर मिलता है और (α, β, γ), त्रिभुज ABC का केन्द्रक है, तो समतल का समीकरण है–

(A*) $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=3$ (B) $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1$ (C) $\frac{3x}{\alpha}+\frac{3y}{\beta}+\frac{3z}{\gamma}=1$ (D) $\alpha x + \beta y + \gamma z = 1$

Sol. 

Let the equation of plane be
माना समतल का समीकरण है —

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

as (α, β, γ) is centroid
जैसाकि (α, β, γ) केन्द्रक है —

$\because$ $\alpha = \frac{a}{3}$; $\beta = \frac{b}{3}$ and और $\gamma = \frac{c}{3}$

$\therefore$ equation of plane $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=3$ $\therefore$ समतल का समीकरण $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=3$

20. Equation of plane which passes through the point of intersection of lines $\frac{x-1}{3}=\frac{y-2}{1}=\frac{z-3}{2}$ and $\frac{x-3}{1}=\frac{y-1}{2}=\frac{z-2}{3}$ and at greatest distance from the point (0, 0, 0) is:

(A) $4x + 3y + 5z = 25$ (B*) $4x + 3y + 5z = 50$
(C) $3x + 4y + 5z = 49$ (D) $x + 7y - 5z = 2$

रेखाओं $\frac{x-1}{3}=\frac{y-2}{1}=\frac{z-3}{2}$ और $\frac{x-3}{1}=\frac{y-1}{2}=\frac{z-2}{3}$ के प्रतिच्छेद बिन्दु से गुजरने वाले और बिन्दु (0, 0, 0) से अधिकतम दूरी पर स्थित समतल का समीकरण है –

(A) $4x + 3y + 5z = 25$ (B) $4x + 3y + 5z = 50$
(C) $3x + 4y + 5z = 49$ (D) $x + 7y - 5z = 2$

Sol. $\frac{x-1}{3}=\frac{y-2}{1}=\frac{z-3}{2}=r$ ... (1) $\frac{x-3}{1}=\frac{y-1}{2}=\frac{z-2}{3}$ .... (2)

$\therefore$ coordinates of any point P on line (1)

$\therefore$ P(3r + 1, r + 2, 2r + 3) for point of intersection of (1) and (2)

$$\frac{3r+1-3}{1}=\frac{r+2-1}{2}=\frac{2r+3-2}{3} \Rightarrow \frac{3r-2}{1}=\frac{r+1}{2}=\frac{2r+1}{3} \quad \therefore \quad r = 1$$

$\therefore$ point of intersection is (4, 3, 5) $\therefore$ the equation of required plane

$\Rightarrow$ $4(x - 4) + 3(y - 3) + 5(z - 5) = 0 \Rightarrow 4x + 3y + 52 = 50$

Hindi. $\frac{x-1}{3} = \frac{y-2}{1} = \frac{z-3}{2} = r$ ... (1) $\frac{x-3}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ .... (2)

$\therefore$ रेखा (1) पर स्थित किसी बिन्दु P के निर्देशांक हैं —

$\therefore$ $P(3r + 1, r + 2, 2r + 3)$

(1) और (2) के प्रतिच्छेद बिन्दु के लिए

$\frac{3r+1-3}{1} = \frac{r+2-1}{2} = \frac{2r+3-2}{3} \Rightarrow \frac{3r-2}{1} = \frac{r+1}{2} = \frac{2r+1}{3}$ $\therefore$ $r = 1$

$\therefore$ प्रतिच्छेद बिन्दु (4, 3, 5) है।

$\therefore$ अभीष्ट समतल का समीकरण है

$4(x - 4) + 3(y - 3) + 5(z - 5) = 0 \Rightarrow 4x + 3y + 52 = 50$

21. The non zero value of 'a' for which the lines $2x - y + 3z + 4 = 0 = ax + y - z + 2$ and $x - 3y + z = 0 = x + 2y + z + 1$ are co-planar is :

(A*) – 2 (B) 4 (C) 6 (D) 0

'a' का अशून्य मान जिसके लिए रेखाएँ $2x - y + 3z + 4 = 0 = ax + y - z + 2$ और $x - 3y + z = 0 = x + 2y + z + 1$ समतलीय हैं, होगा —

(A) – 2 (B) 4 (C) 6 (D) 0

Sol. $2x - y + 3z + 4 = 0 = ax + y - z + 2$ ... (1)

$\because$ equation of plane through (1) is $(2x - y + 3z + 4) + \lambda(ax + y - z + 2) = 0$

$x(2 + a\lambda) + y(\lambda - 1) + z(3 - \lambda) + (4 + 2\lambda) = 0$ .... (2)

$x - 3y + z = 0 = x + 2y + 2 + 1$ ... (3)

$\because$ equation of plane passing through (3) is

$(x - 3y + z) + \mu(x + 2y + z + 1) = 0 \Rightarrow x(1 + \mu) + y(2\mu - 3) + z(\mu + 1) + \mu = 0$ .... (4)

if lines (1) and (3) are coplanar, then $\frac{2+a\lambda}{\mu+1} = \frac{\lambda-1}{2\mu-3} = \frac{3-\lambda}{\mu+1} = \frac{4+2\lambda}{\mu}$

Solving this we get $\lambda = -1$, $\mu = 1$ $\therefore$ $a = -2$

Hindi. $2x - y + 3z + 4 = 0 = ax + y - z + 2$ ... (1)

$\because$ (1) से गुजरने वाले समतल का समीकरण है —

$(2x - y + 3z + 4) + \lambda(ax + y - z + 2) = 0$

$x(2 + a\lambda) + y(\lambda - 1) + z(3 - \lambda) + (4 + 2\lambda) = 0$ .... (2)

$x - 3y + z = 0 = x + 2y + 2 + 1$ ... (3)

$\because$ (3) से गुजरने वाले समतल का समीकरण है —

$(x - 3y + z) + \mu(x + 2y + z + 1) = 0 \Rightarrow x(1 + \mu) + y(2\mu - 3) + z(\mu + 1) + \mu = 0$ .... (4)

यदि रेखा (1) और (3) समतलीय हैं, तब $\frac{2+a\lambda}{\mu+1} = \frac{\lambda-1}{2\mu-3} = \frac{3-\lambda}{\mu+1} = \frac{4+2\lambda}{\mu}$

इनको हल करने पर $\lambda = -1$, $\mu = 1$ $\therefore$ $a = -2$

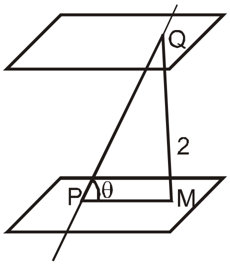
22. A line having direction ratios 3, 4, 5 cuts 2 planes $2x - 3y + 6z - 12 = 0$ and $2x - 3y + 6z + 2 = 0$ at point P & Q, then find length of PQ

(A*) $\frac{35\sqrt{2}}{12}$ (B) $\frac{35\sqrt{2}}{24}$ (C) $\frac{35\sqrt{2}}{6}$ (D) $\frac{35\sqrt{2}}{8}$

3, 4, 5 दिक् अनुपात वाली एक सरल रेखा दो समतलों $2x - 3y + 6z - 12 = 0$ एवं $2x - 3y + 6z + 2 = 0$ को बिन्दुओं P एवं Q, पर प्रतिच्छेद करती है, तो PQ की लम्बाई है-

(A*) $\frac{35\sqrt{2}}{12}$ (B) $\frac{35\sqrt{2}}{24}$ (C) $\frac{35\sqrt{2}}{6}$ (D) $\frac{35\sqrt{2}}{8}$

Sol.



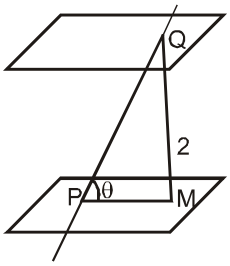
distance between plane = $\left|\dfrac{2+12}{7}\right| = 2$

L between line and plane = θ

$\sin\theta = \dfrac{6-12+30}{5\sqrt{2}.7} = \dfrac{24}{35\sqrt{2}}$

$\sin\theta = \dfrac{2}{PQ} \Rightarrow PQ = 2 \times \dfrac{35\sqrt{2}}{24} = \dfrac{35\sqrt{2}}{12}$

Hindi



समतलों के मध्य दूरी = $\left|\dfrac{2+12}{7}\right| = 2$

L रेखा और समतल के मध्य कोण = θ

$\sin\theta = \dfrac{6-12+30}{5\sqrt{2}.7} = \dfrac{24}{35\sqrt{2}}$

$\sin\theta = \dfrac{2}{PQ} \Rightarrow PQ = 2 \times \dfrac{35\sqrt{2}}{24} = \dfrac{35\sqrt{2}}{12}$

23. A line L_1 having dierection ratios 1, 0, 1 lies on xz plane. Now this xz plane is rotated about z-axis by an angle of 90°. Now the new position of L_1 is L_2. The angle between L_1 & L_2 is : **[16JM120037]**

एक रेखा L_1 के दिक् अनुपात 1, 0, 1 है जो xz समतल पर स्थित है। अब इस xz समतल को z-अक्ष के सापेक्ष 90° कोण से घूर्णन कराया जाता है। इस प्रकार L_1 की नयी स्थिति L_2 है, तो L_1 एवं L_2 के मध्य कोण होगा–

(A) 30° (B*) 60° (C) 90° (D) 45°

Sol. $L_1 : 1, 0, 1 \Rightarrow \quad \alpha = 45°, \beta = 90°, \gamma = 45° \Rightarrow \ell_1 = \dfrac{1}{\sqrt{2}}, m_1 = 0, n_1 - \dfrac{1}{\sqrt{2}}$

$L_2 : \alpha = 90°, \beta = 45°, \gamma = 45°$

So अतः $\ell_1 = 0, m_1 = \dfrac{1}{\sqrt{2}}, n_1 = \dfrac{1}{\sqrt{2}}$

Let angle between L_1 & L_2 be θ. माना L_1 तथा L_2 के मध्य कोण θ है तब

so अतः $\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2 \Rightarrow \cos\theta = \dfrac{1}{2}, \theta = 60°$

24._ Focii of ellipse lie on plane 2x + 3y + 6z = 0 are (54, 18, –27) and (–54, –18, 27). If eccentricity of ellipse is $\dfrac{3}{5}$ then find equation of directrices .

समतल $2x + 3y + 6z = 0$ पर स्थित दीर्घवृत्त की नाभियां (54, 18, –27) या (–54, –18, 27) यदि दीर्घवृत्त की उत्केन्द्रता $\frac{3}{5}$ है तब नियताओं के समीकरण है–

(A*) $\frac{x-150}{3} = \frac{y-50}{-6} = \frac{z+75}{2}, \frac{x+150}{3} = \frac{y+50}{-6} = \frac{z-75}{2}$

(B) $\frac{x-150}{-3} = \frac{y+50}{-6} = \frac{z+75}{2}, \frac{x+150}{-3} = \frac{y-50}{-6} = \frac{z-75}{2}$

(C) $\frac{x-150}{3} = \frac{y+50}{-6} = \frac{z+75}{2}, \frac{x+150}{3} = \frac{y-50}{-6} = \frac{z-75}{2}$

(D) $\frac{x-150}{3} = \frac{y-50}{-6} = \frac{z-75}{-2}, \frac{x+150}{3} = \frac{y+50}{-6} = \frac{z+75}{-2}$

Sol. Equation of major axis of Ellipse is $\frac{x}{6} = \frac{y}{2} = \frac{z}{-3}$ and centre is (0, 0, 0)

D.Rs of directrices are components of $(6\hat{i} + 2\hat{j} - 3\hat{k}) \times (2\hat{i} + 3\hat{j} + 6\hat{k}) \Rightarrow (3, -6, 2)$

point of intersection of directrices and major axis are (150, 50, –75) and (–150, –50, 75)

$\Rightarrow$ equation of directories are

$\frac{x-150}{3} = \frac{y-50}{-6} = \frac{z+75}{2}$ and $\frac{x+150}{3} = \frac{y+50}{-6} = \frac{z-75}{2}$

Hindi. दीर्घवृत्त के दीर्घ अक्ष का समीकरण $\frac{x}{6} = \frac{y}{2} = \frac{z}{-3}$ तथा केन्द्र (0, 0, 0) है।

$(6\hat{i} + 2\hat{j} - 3\hat{k}) \times (2\hat{i} + 3\hat{j} + 6\hat{k}) \Rightarrow (3, -6, 2)$ के नियताओं के दिक्कोज्याएं

नियता और दीर्घअक्ष (150, 50, –75) और (–150, –50, 75) के प्रतिच्छेद बिन्दु $\Rightarrow$ नियताओं के समीकरण

$\frac{x-150}{3} = \frac{y-50}{-6} = \frac{z+75}{2}$ और $\frac{x+150}{3} = \frac{y+50}{-6} = \frac{z-75}{2}$ है।

PART-II: NUMERICAL VALUE QUESTIONS

भाग-II : संख्यात्मक प्रश्न (NUMERICAL VALUE QUESTIONS)

INSTRUCTION :

❖ The answer to each question is **NUMERICAL VALUE** with two digit integer and decimal upto two digit.

❖ If the numerical value has more than two decimal places **truncate/round-off** the value to **TWO** decimal placed.

निर्देश :

❖ इस खण्ड में प्रत्येक प्रश्न का उत्तर **संख्यात्मक मान** के रूप में है जिसमें दो पूर्णांक अंक तथा दो अंक दशमलव के बाद में है।

❖ यदि संख्यात्मक मान में दो से अधिक दशमलव स्थान है, तो संख्यात्मक मान को दशमलव के **दो** स्थानों तक **ट्रंकेट/राउंड ऑफ (truncate/round-off)** करें।

1. Given $f^2(x) + g^2(x) + h^2(x) \le 9$ and $U(x) = 3f(x) + 4g(x) + 10h(x)$, where f(x), g(x) and h(x) are continuous $\forall\ x \in R$ then maximum value of U(x) is.

दिया गया है कि $f^2(x) + g^2(x) + h^2(x) \le 9$ तथा $U(x) = 3f(x) + 4g(x) + 10h(x)$ जहाँ f(x), g(x) तथा h(x) सभी $x \in R$ के लिए सतत् है, तब U(x) का अधिकतम मान है –

Ans. **33.54**

Sol. $\max \{U(x)\} = \max. \{(3\hat{i}+4\hat{j}+10\hat{k}) \cdot \left(f(x)\hat{i} + g(x)\hat{j} + h(x)\hat{k}\right)\}$

$= \max. \left\{\sqrt{125} \cdot \sqrt{f(x)^2+g(x)^2+f(x)^2} \cdot \cos\theta\right\} = \sqrt{1125} \Rightarrow \quad N = 1125 \Rightarrow \quad N - 1000 = 125$

2. If in a plane $A_1, A_2, A_3, \ldots\ldots A_{20}$ are the vertices of a regular polygon having 20 sides and O is its centre and $\sum_{i=1}^{19}(\overrightarrow{OA_i} \times \overrightarrow{OA_{i+1}}) = \lambda(\overrightarrow{OA_2} \times \overrightarrow{OA_1})$ then $|\lambda|$ is **[16JM120040]**

यदि समतल में $A_1, A_2, A_3, \ldots\ldots A_{20}$ समबहुभुज के शीर्ष है जिसकी 20 भुजाएँ है और O इसका केन्द्र है तथा $\sum_{i=1}^{19}(\overrightarrow{OA_i} \times \overrightarrow{OA_{i+1}}) = \lambda(\overrightarrow{OA_2} \times \overrightarrow{OA_1})$ तब $|\lambda|$ का मान है

Ans. **19.00**

Sol. $\sum_{i=1}^{n-1}\overrightarrow{OA_i} \times \overrightarrow{OA_{i+1}} = \overrightarrow{OA_1} \times \overrightarrow{OA_2} + \overrightarrow{OA_2} \times \overrightarrow{OA_3} \ldots\ldots\ldots + \overrightarrow{OA_{n-1}} \times \overrightarrow{OA_n} = (n-1)\ (\overrightarrow{OA_1} \times \overrightarrow{OA_2})$

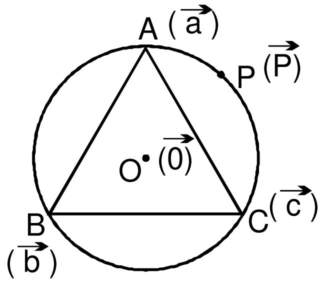
$= (1-n)\ (\overrightarrow{OA_2} \times \overrightarrow{OA_1})$ when जब $\quad n = 20 \Rightarrow \quad \lambda = -19$

3. In an equilateral ΔABC find the value of $\dfrac{|\overrightarrow{PA}|^2 + |\overrightarrow{PB}|^2 + |\overrightarrow{PC}|^2}{R^2}$ where P is any arbitrary point lying on its circumcircle, is

समबाहु ΔABC में $\dfrac{|\overrightarrow{PA}|^2 + |\overrightarrow{PB}|^2 + |\overrightarrow{PC}|^2}{R^2}$ का मान ज्ञात कीजिए जहां इसके परिगत वृत्त पर स्वेच्छ बिन्दु P है–

Ans. **06.00**

Sol.



Let $O(\vec{0})$ be the circumcentre of ΔABC

माना $O(\vec{0})$, ΔABC का परिकेन्द्र है।

Given $\quad (\vec{a}) = |\vec{b}| = |\vec{c}| = R \Rightarrow 0 = \dfrac{\vec{a}+\vec{b}+\vec{c}}{3} \Rightarrow \vec{a}+\vec{b}+\vec{c} = 0$

$\dfrac{|\overrightarrow{PA}|^2 + |\overrightarrow{PB}|^2 + |\overrightarrow{PC}|^2}{R^2} = ? \Rightarrow \dfrac{|\vec{P}-\vec{a}|^2 + |\vec{P}-\vec{b}|^2 + |\vec{P}-\vec{c}|^2}{R^2}$

$\dfrac{3|\vec{P}| + |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 - 2\vec{P}.(\vec{a}+\vec{b}+\vec{c})}{R^2} \quad ; \quad \dfrac{6R^2}{R^2} = 6$

4. Let $\vec{A} = 2\hat{i} + \hat{k}$, $\vec{B} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{C} = 4\hat{i} - 3\hat{j} + 7\hat{k}$. If a vector $\vec{R} = \alpha\hat{i} - \beta\hat{j} + \gamma\hat{k}$ satisfies $\vec{R} \times \vec{B} = \vec{C} \times \vec{B}$ and $\vec{R} \cdot \vec{A} = 0$ then $\alpha^2 + \beta^2 + \gamma^2$ is **[16JM120039]**

माना $\vec{A} = 2\hat{i} + \hat{k}$, $\vec{B} = \hat{i} + \hat{j} + \hat{k}$ तथा $\vec{C} = 4\hat{i} - 3\hat{j} + 7\hat{k}$ है। यदि सदिश $\vec{R} = \alpha\hat{i} - \beta\hat{j} + \gamma\hat{k}$ satisfies $\vec{R} \times \vec{B} = \vec{C} \times \vec{B}$ और $\vec{R} \cdot \vec{A} = 0$ को सन्तुष्ट करती है तब $\alpha^2 + \beta^2 + \gamma^2$ is

Ans. **69.00**

Sol. $(\vec{R} - \vec{C}) \times \vec{B} = \vec{O} \Rightarrow \quad \vec{R} = \vec{C} + \lambda\vec{B} \quad \Rightarrow \quad A.C + \lambda A.B = 0 \Rightarrow \quad 15 + 3\lambda = 0$

$\Rightarrow \quad \lambda = -5 \Rightarrow \quad \vec{R} = -\hat{i} - 8\hat{j} + 2\hat{k}$

5. A line $\frac{x+2}{1} = \frac{y-3}{2} = \frac{z-k}{3}$ cuts the y-z plane and the x-y plane at A and B respectively. If $\angle AOB = \frac{\pi}{2}$, then k, where O is the origin, is

एक रेखा $\frac{x+2}{1} = \frac{y-3}{2} = \frac{z-k}{3}$, y-z समतल तथा x-y समतल को क्रमशः A और B पर प्रतिच्छेद करती है यदि $\angle AOB = \frac{\pi}{2}$ तब k का मान है जहाँ O मूल बिन्दु है–

Ans. **04.50**

Sol. $\frac{x+2}{1} = \frac{y-3}{2} = \frac{z-k}{3} = \lambda \quad \Rightarrow \quad (\lambda - 2, 2\lambda + 3, 3\lambda + k)$ for A, $\lambda = 2$

A(0, 7, 6 + k) $\quad \Rightarrow \quad$ for B $\lambda = -\frac{k}{3} \quad \Rightarrow \quad B\left(-2 - \frac{k}{3}, \ 3 - \frac{2k}{3}, \ 0\right)$

$\angle AOB = 90° \quad \Rightarrow \quad \overrightarrow{AO} \cdot \overrightarrow{OB} = 0 \quad \Rightarrow \quad 7\left(-\ 3 + \frac{2k}{3}\right) = 0$ or $k = \frac{9}{2} \Rightarrow 2k = 9$

Hindi $\frac{x+2}{1} = \frac{y-3}{2} = \frac{z-k}{3} = \lambda \quad \Rightarrow \quad (\lambda - 2, 2\lambda + 3, 3\lambda + k)$ A के लिए $\lambda = 2$

A(0, 7, 6 + k) $\Rightarrow$ B के लिए $\lambda = -\frac{k}{3} \quad \Rightarrow \quad B\left(-2 - \frac{k}{3}, \ 3 - \frac{2k}{3}, \ 0\right)$

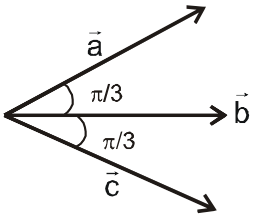
$\angle AOB = 90° \quad \Rightarrow \quad \overrightarrow{AO} \cdot \overrightarrow{OB} = 0 \quad \Rightarrow \quad 7\left(-\ 3 + \frac{2k}{3}\right) = 0$ या $k = \frac{9}{2} \Rightarrow 2k = 9$

6. Given four non zero vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$. The vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar but not collinear pair by pair and vector $\vec{d}$ is not coplanar with vectors $\vec{a}, \vec{b}$ and $\vec{c}$ and $(\vec{a} \wedge \vec{b}) = (\vec{b} \wedge \vec{c}) = \frac{\pi}{3}$, $(\vec{d} \wedge \vec{a}) = \alpha$ and $(\vec{d} \wedge \vec{b}) = \beta$, if $(\vec{d} \wedge \vec{c}) = \cos^{-1}(m\cos\beta + n\cos\alpha)$ then m – n is :

दिया है कि $\vec{a}, \vec{b}, \vec{c}$ एवं $\vec{d}$ चार अशून्य सदिश हैं। सदिश $\vec{a}, \vec{b}$ एवं $\vec{c}$ समतलीय हैं लेकिन युग्म में असंरेखीय हैं और सदिश $\vec{d}$ सदिशों $\vec{a}, \vec{b}$ एवं $\vec{c}$ के साथ समतलीय नहीं हैं और $(\vec{a} \wedge \vec{b}) = (\vec{b} \wedge \vec{c}) = \frac{\pi}{3}$, $(\vec{d} \wedge \vec{a}) = \alpha$, $(\vec{d} \wedge \vec{b}) = \beta$ यदि $(\vec{d} \wedge \vec{c}) = \cos^{-1}(m\cos\beta + n\cos\alpha)$, तब m – n है

Ans. **02.00**

Sol.



Angle between vector $\vec{a}$ & $\vec{b}$ remains same even if we presume them as unit vector. Here for sake of convinience let $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are unit vectors.

$\vec{a}.\vec{b} = \cos\frac{\pi}{3} = \frac{1}{2}$ ....(1) ; $\vec{b}.\vec{c} = \frac{1}{2}$ ....(2)

$\vec{a} \ . \ \vec{d} = \cos\alpha$ ....(3) ; $\vec{b}.\vec{d} = \cos\beta$ ....(4)

also $\vec{b} = \lambda\ (\vec{a}+\vec{c})$

Since $\vec{b}$ is presumed as unit vector

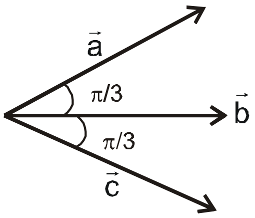
$|\lambda(\vec{a}+\vec{c})\ | = 1 \ \Rightarrow\ \lambda^2(\vec{a}^2+\vec{c}^2+2\vec{a}.\vec{c}) = 1$

or $\lambda^2(1+1-1) = 1 \ \Rightarrow\ \lambda = 1$

$\therefore$ $\vec{b} = (\vec{a}+\vec{c}) \ \Rightarrow\ \vec{c} = \vec{b}-\vec{a}$

again $\vec{d}.\vec{c} = |\vec{d}||\vec{c}|\cos\theta = \vec{d}.(\vec{b}-\vec{a})$ $\cos\theta = \cos\beta - \cos\alpha$ $\theta = \cos^{-1}(\cos\beta - \cos\alpha)$

Hindi



यदि सदिश $\vec{a}$ और $\vec{b}$ को इकाई सदिश मान लिया जाये तब भी इनके मध्य बनने वाला कोण अपरिवर्तित रहता है। अतः सुविधा के लिये $\vec{a}$, $\vec{b}$, $\vec{c}$ एवं $\vec{d}$ को हम इकाई सदिश मान लेते है।

$\vec{a}.\vec{b} = \cos\frac{\pi}{3} = \frac{1}{2}$ ....(1) ; $\vec{b}\ .\ \vec{c} = \frac{1}{2}$ ....(2)

$\vec{a}.\vec{d} = \cos\alpha$ ....(3) ; $\vec{b}.\vec{d} = \cos\beta$ ....(4)

और $\vec{b} = \lambda\,(\vec{a}+\vec{c})$

जैसा कि सदिश $\vec{b}$ को इकाई सदिश मान लिया है

$|\lambda(\vec{a}+\vec{c})\ | = 1 \ \Rightarrow\ \lambda^2(\vec{a}^2+\vec{c}^2+2\vec{a}.\vec{c}) = 1$

या $\lambda^2(1+1-1) = 1 \ \Rightarrow\ \lambda = 1$

$\therefore$ $\vec{b} = (\vec{a}+\vec{c}) \ \Rightarrow\ \vec{c} = \vec{b}-\vec{a}$

पुनः $\vec{d}.\vec{c} = |\vec{d}||\vec{c}|\cos\theta = \vec{d}.(\vec{b}-\vec{a})\cos\theta = \cos\beta - \cos\alpha \ \Rightarrow\ \theta = \cos^{-1}(\cos\beta - \cos\alpha)$

7. If the circumcentre of the tetrahedron OABC is given by $\frac{\vec{a}^2\left(\vec{b}\times\vec{c}\right)+\vec{b}^2\left(\vec{c}\times\vec{a}\right)+\vec{c}^2\left(\vec{a}\times\vec{b}\right)}{\alpha}$, where $\vec{a}, \vec{b}$ & $\vec{c}$ are the position vectors of the points A, B, C respectively relative to the origin 'O' such that $[\vec{a}\ \vec{b}\ \vec{c}] = 36$ then α is **[16JM120041]**

यदि चतुष्फलक OABC का परिकेन्द्र $\frac{\vec{a}^2\left(\vec{b}\times\vec{c}\right)+\vec{b}^2\left(\vec{c}\times\vec{a}\right)+\vec{c}^2\left(\vec{a}\times\vec{b}\right)}{\alpha}$ दिया गया है, जहाँ मूल बिन्दु के सापेक्ष $\vec{a}, \vec{b}$ तथा $\vec{c}$ क्रमशः बिन्दु A, B, C के स्थिति सदिश है जबकि $[\vec{a}\ \vec{b}\ \vec{c}] = 36$ तब α है –

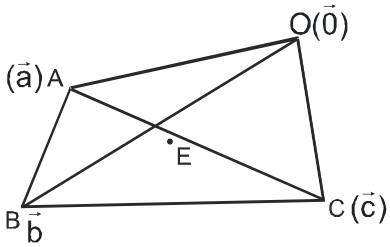
Ans. **72.00**

Sol. vectors $\vec{a}$, $\vec{b}$ & $\vec{c}$ are non coplanar so are the vectors $\vec{a}\times\vec{b}$, $\vec{b}\times\vec{c}$

Let position vector of circumcentre $\vec{r} \equiv x(\vec{a}\times\vec{b}) + y(\vec{b}\times\vec{c}) + z(\vec{c}\times\vec{a})$

also OE = AE = EB = EC $\Rightarrow$ $|\vec{r}| = |\vec{r}-\vec{a}| = |\vec{r}-\vec{b}| = |\vec{r}-\vec{c}|$

or $\vec{r}^2 = \vec{r}^2 + \vec{a}^2 - 2\vec{r}.\vec{a} = \vec{r}^2 + \vec{b}^2 - 2\vec{r}.\vec{b} = \vec{r}^2 + \vec{c}^2 - 2\vec{r}.\vec{c}^2$



$\Rightarrow$ $2\vec{r}.\vec{a} = \vec{a}^2$, $2\vec{r}.\vec{b} = \vec{b}^2$, or $2y[\vec{a}\ \ \vec{b}\ \ \vec{c}] = \vec{a}^2$ $\Rightarrow$ $y = \dfrac{\vec{a}^2}{2[\vec{a}\ \ \vec{b}\ \ \vec{c}]}$

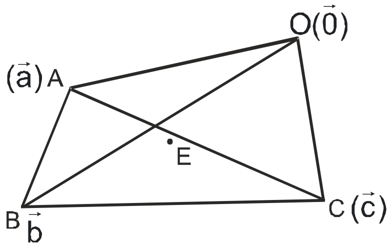
Similarly z & x can be obtained

Hindi चूँकि $\vec{a}$, $\vec{b}$ एंव $\vec{c}$ असमतलीय हैं, तो सदिश $\vec{a}\times\vec{b}$, $\vec{b}\times\vec{c}$ भी असमतलीय होगें।

माना परिकेन्द्र का स्थित सदिश $\vec{r} \equiv x(\vec{a}\times\vec{b}) + y(\vec{b}\times\vec{c}) + z(\vec{c}\times\vec{a})$ है।

एवं OE = AE = EB = EC $\Rightarrow$ $|\vec{r}| = |\vec{r}-\vec{a}| = |\vec{r}-\vec{b}| = |\vec{r}-\vec{c}|$

या $\vec{r}^2 = \vec{r}^2 + \vec{a}^2 - 2\vec{r}.\vec{a} = \vec{r}^2 + \vec{b}^2 - 2\vec{r}.\vec{b} = \vec{r}^2 + \vec{c}^2 - 2\vec{r}.\vec{c}^2$



$\Rightarrow$ $2\vec{r}.\vec{a} = \vec{a}^2$, $2\vec{r}.\vec{b} = \vec{b}^2$, या $2y[\vec{a}\ \ \vec{b}\ \ \vec{c}] = \vec{a}^2$ $\Rightarrow$ $y = \dfrac{\vec{a}^2}{2[\vec{a}\ \ \vec{b}\ \ \vec{c}]}$

इसी प्रकार z और x को भी प्राप्त किया जा सकता है।

8. Given three point on x – y plane as O(0, 0), A(1, 0) & B(–1, 0). Point P moving on the given plane satisfying the condition $(\vec{PA}\ .\ \vec{PB}) + 3(\vec{OA}\ .\ \vec{OB}) = 0$ **[16JM120042]**

If the maximum & minimum values of $|\vec{PA}||\vec{PB}|$ is M & m respectively then the value of $M^2 + m^2$ is

Ans. **34.00**

दिए गए तीन बिन्दु O(0, 0) A(1, 0) & B(–1, 0) xy समतल पर है। बिन्दु P समतल पर इस प्रकार गति करता है कि प्रतिबन्ध $(\vec{PA}\ .\ \vec{PB}) + 3(\vec{OA}\ .\ \vec{OB}) = 0$ है। यह $|\vec{PA}||\vec{PB}|$ का अधिकतम व न्यूनतम मान क्रमशः M तथा m है। तब $M^2 + m^2$ का मान है–

Sol. P = xi + yj ; $\vec{AP} = (x-1)i + yj$; $\vec{BP} = (x+1)i + yj$

$\vec{PA}\ .\ \vec{PB} = x^2 - 1 + y^2$; $\vec{OA}\ .\ \vec{OB} = -1$

Now अब $(\vec{PA}.\vec{PB}) + 3(\vec{OA}.\vec{OB}) = 0$ $\Rightarrow$ $x^2 + y^2 - 4 = 0$ $\Rightarrow$ $x^2 + y^2 = 4$

$|\vec{PA}||\vec{PB}| = \sqrt{(x-1)^2 + y^2}\ \sqrt{(x+1)^2 + y^2} = \sqrt{5-2x}\sqrt{5+2x}$

$|\vec{PA}||\vec{PB}| = \sqrt{25-4x^2}$

Now from $x^2 + y^2 = 4$ से put $x = 2\cos\theta$ रखने पर $y = 2\sin\theta$

$|\vec{PA}||\vec{PB}| = \sqrt{25-16\cos^2\theta}$; $|\vec{PA}||\vec{PB}|_{max} = \sqrt{25} = M$

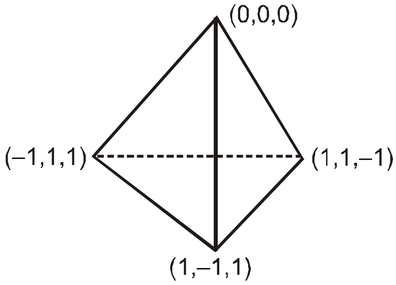
$|\vec{PA}||\vec{PB}|_{min} = \sqrt{9} = m$; $M^2 + m^2 = 25 + 9 = 34$

9. If the volume of tetrahedron formed by planes whose equations are $y + z = 0$, $z + x = 0$, $x + y = 0$ and $x + y + z = 1$ is t cubic unit, then the value of t is

यदि समतलों $y + z = 0$, $z + x = 0$, $x + y = 0$ तथा $x + y + z = 1$ द्वारा बनाये गये चतुष्फलक का आयतन t घन इकाई है तो t का मान ज्ञात कीजिए।

Ans. **00.66 or 00.67**

Sol.



The planes are

$y + z = 0$ ....(1)

$z + x = 0$ ....(2)

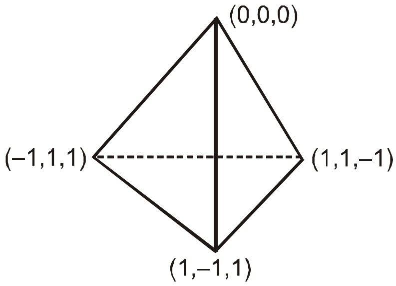
$x + y = 0$ ....(3)

$x + y + z = 1$ ....(4)

Solving above equations we get vertices of the tetrahedron as (0,0,0), (–1,1,1), (1,–1,1) and (1,1,–1)

$$\text{Required volume} = \left| \frac{1}{6}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \right| = \left| \frac{1}{6}\begin{vmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 0 & 0 & -1 \end{vmatrix} \right| = \frac{4}{6} = \frac{2}{3} \Rightarrow t = \frac{2}{3} \Rightarrow 3t = 2$$

Hindi



समतल है

$y + z = 0$ ....(1)

$z + x = 0$ ....(2)

$x + y = 0$ ....(3)

$x + y + z = 1$ ....(4)

उपरोक्त समीकरणों को हल करने पर हमें चतुष्फलक के शीर्ष (0,0,0), (–1,1,1), (1,–1,1) तथा (1,1,–1) मिलते हैं

$$\text{अभीष्ट आयतन} = \left| \frac{1}{6}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \right| = \left| \frac{1}{6}\begin{vmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 0 & 0 & -1 \end{vmatrix} \right| = \frac{4}{6} = \frac{2}{3} \Rightarrow t = \frac{2}{3} \Rightarrow 3t = 2$$

10. If $\vec{r}$ represents the position vector of point R in which the line AB cuts the plane CDE, where position vectors of points A, B, C, D, E are respectively $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} + 2\hat{k}$, $\vec{c} = -4\hat{j} + 4\hat{k}$, $\vec{d} = 2\hat{i} - 2\hat{j} + 2\hat{k}$ and $\vec{e} = 4\hat{i} + \hat{j} + 2\hat{k}$, then $|\vec{r}|$ is : **[16JM120043]**

यदि रेखा AB तथा समतल CDE के प्रतिच्छेद बिन्दु R का स्थिति सदिश $\vec{r}$ है, जहाँ बिन्दुओं A, B, C, D, E के स्थिति सदिश क्रमशः $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} + 2\hat{k}$, $\vec{c} = -4\hat{j} + 4\hat{k}$, $\vec{d} = 2\hat{i} - 2\hat{j} + 2\hat{k}$ और $\vec{e} = 4\hat{i} + \hat{j} + 2\hat{k}$, है, तब $|\vec{r}|$ है:

Ans. **04.24**

Sol. Equation of line AB is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}) \quad \Rightarrow \quad \vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ ....(1)

$\overrightarrow{CD} = 2\hat{i} + 2\hat{j} - 2\hat{k}$; $\overrightarrow{CE} = 4\hat{i} + 5\hat{j} - 2\hat{k}$; $\vec{n} = \overrightarrow{CD} \times \overrightarrow{CE} = 6\hat{i} - 4\hat{j} + 2\hat{k}$

So equation of plane CDE is $3x - 2y + z = 12$. Solve with line $\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$

$3(1+\lambda) - 2(2-\lambda) + 1 + \lambda = 12 \quad \Rightarrow \quad \lambda = 2$ Hence R is $3\hat{i} + 3\hat{k}$

Hindi रेखा AB का समीकरण $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}) \quad \Rightarrow \quad \vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ ....(1)

$\overrightarrow{CD} = 2\hat{i} + 2\hat{j} - 2\hat{k}$; $\overrightarrow{CE} = 4\hat{i} + 5\hat{j} - 2\hat{k}$; $\vec{n} = \overrightarrow{CD} \times \overrightarrow{CE} = 6\hat{i} - 4\hat{j} + 2\hat{k}$

समतल CDE का समीकरण $3x - 2y + z = 12$ है –

रेखा $\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ के साथ हल करने पर $3(1+\lambda) - 2(2-\lambda) + 1 + \lambda = 12 \Rightarrow \lambda = 2$

अतः बिन्दु R, $3\hat{i} + 3\hat{k}$ है।

11. Line L_1 is parallel to vector $\vec{\alpha} = -3\hat{i} + 2\hat{j} + 4\hat{k}$ and passes through a point A(7, 6, 2) and line L_2 is parallel to a vector $\vec{\beta} = 2\hat{i} + \hat{j} + 3\hat{k}$ and passes through a point B(5, 3, 4). Now a line L_3 parallel to a vector $\vec{r} = 2\hat{i} - 2\hat{j} - \hat{k}$ intersects the lines L_1 and L_2 at points C and D respectively, then $|4\overrightarrow{CD}|$ is equal to :

Ans. **36.00** **[16JM120044]**

रेखा L_1, सदिश $\vec{\alpha} = -3\hat{i} + 2\hat{j} + 4\hat{k}$ के समान्तर है और बिन्दु A(7, 6, 2) से गुजरती है और रेखा L_2 सदिश $\vec{\beta} = 2\hat{i} + \hat{j} + 3\hat{k}$ के समान्तर है और एक बिन्दु B(5, 3, 4) से गुजरती है। अब एक रेखा L_3 एक सदिश $\vec{r} = 2\hat{i} - 2\hat{j} - \hat{k}$ के समान्तर है और रेखाओं L_1 तथा L_2 को क्रमशः बिन्दुओं C और D पर प्रतिच्छेद करती है, तो $|4\overrightarrow{CD}|$ बराबर है :

Sol. Equation of line L_1 is $7\hat{i} + 6\hat{j} + 2\hat{k} + \lambda(-3\hat{i} + 2\hat{j} + 4\hat{k})$

Equation of line L_2 is $5\hat{i} + 3\hat{j} + 4\hat{k} + \mu(2\hat{i} + \hat{j} + 3\hat{k})$

$\overrightarrow{CD} = 2\hat{i} + 3\hat{j} - 2\hat{k} + \lambda(-3\hat{i} + 2\hat{j} + 4\hat{k}) - \mu(2\hat{i} + \hat{j} + 3\hat{k})$. since it is parallel to $2\hat{i} - 2\hat{j} - \hat{k}$

$$\therefore \quad \frac{2 - 3\lambda - 2\mu}{2} = \frac{3 + 2\lambda - \mu}{-2} = \frac{-2 + 4\lambda - 3\mu}{-1} \qquad \therefore \quad \lambda = 2, \quad \mu = 1$$

$$\therefore \quad \overrightarrow{CD} = -6\hat{i} + 6\hat{j} + 3\hat{k} \qquad \therefore \quad |4\overrightarrow{CD}| = 36$$

Hindi. रेखा L_1 का समीकरण $7\hat{i} + 6\hat{j} + 2\hat{k} + \lambda(-3\hat{i} + 2\hat{j} + 4\hat{k})$ है।

रेखा L_2 का समीकरण $5\hat{i} + 3\hat{j} + 4\hat{k} + \mu(2\hat{i} + \hat{j} + 3\hat{k})$ है।

$\overrightarrow{CD} = 2\hat{i} + 3\hat{j} - 2\hat{k} + \lambda(-3\hat{i} + 2\hat{j} + 4\hat{k}) - \mu(2\hat{i} + \hat{j} + 3\hat{k})$

चूंकि यह $2\hat{i} - 2\hat{j} - \hat{k}$ के समान्तर है।

$$\therefore \quad \frac{2 - 3\lambda - 2\mu}{2} = \frac{3 + 2\lambda - \mu}{-2} = \frac{-2 + 4\lambda - 3\mu}{-1} \qquad \therefore \quad \lambda = 2, \quad \mu = 1$$

$$\therefore \quad \overrightarrow{CD} = -6\hat{i} + 6\hat{j} + 3\hat{k} \qquad \therefore \quad |4\overrightarrow{CD}| = 36$$

12. L is the equation of the straight line which passes through the point (2, –1, –1); is parallel to the plane $4x + y + z + 2 = 0$ and is perpendicular to the line of intersection of the planes $2x + y = 0 = x - y + z$. If the point $(3, \alpha, \beta)$ lies on line L, then absolute value of $\frac{\alpha}{\beta}$ is

माना सरल रेखा का समीकरण जो बिन्दु (2, –1, –1) से गुजरता है तथा समतल $4x + y + z + 2 = 0$ के समान्तर है और समतलों $2x + y = 0 = x - y + z$ की प्रतिच्छेदन रेखा के लम्बवत् है यदि बिन्दु $(3, \alpha, \beta)$ रेखा L पर स्थित है तब $\frac{\alpha}{\beta}$ का निरपेक्ष मान होगा –

Ans. **01.75**

Sol. $\frac{x-2}{a} = \frac{y+1}{b} = \frac{z+1}{c} \quad \Rightarrow \quad 4a + b + c = 0 \quad$ ...(i)

$2x + y = 0 = x - y + z \quad \Rightarrow \quad \begin{vmatrix} \hat{i} & \hat{i} & \hat{k} \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = \hat{i}(1-0) - \hat{j}(2-0) + \hat{k}(-2-1) = \hat{i} - 2\hat{j} - 3\hat{k}$

$a - 2b - 3c = 0 \quad$ ... (ii)

From (i) & (ii) (i) और (ii) से

$4a + b + c = 0 \quad \Rightarrow \quad a - 2b - 3c = 0 \Rightarrow \frac{a}{-3+2} = \frac{b}{1+12} = \frac{c}{-8-1} \quad \Rightarrow \frac{a}{-1} = \frac{b}{13} = \frac{c}{-9}$

$\therefore$ equation of the line $\quad \therefore$ रेखा का समीकरण

$\frac{x-2}{-1} = \frac{y+1}{13} = \frac{z+1}{-9} \Rightarrow \frac{3-2}{-1} = \frac{\alpha+1}{13} = \frac{\beta+1}{-9} \Rightarrow \alpha = -14$ and $\beta = 8 \Rightarrow |\alpha + \beta| = 6.$

13. The lines $\frac{x+4}{3} = \frac{y+6}{5} = \frac{z-1}{-2}$ and $3x - 2y + z + 5 = 0 = 4x + 3y - 4z - 3k$ are coplanar, then the value of k is

रेखाएँ $\frac{x+4}{3} = \frac{y+6}{5} = \frac{z-1}{-2}$ तथा $3x - 2y + z + 5 = 0 = 4x + 3y - 4z - 3k$ समतलीय हैं, तब k का मान है

Ans. **10.66 or 10.67**

Sol. $L_1 : \frac{x+4}{3} = \frac{y+6}{5} = \frac{z-1}{-2} = r \quad ; \quad L_2 : 3x - 2y + z + 5 = 0 = 4x + 3y - 4z - 3k$

Any point on the first line is $(3r - 4, 5r - 6, -2r + 1)$

As lines are coplanar therefore this point must lie on both the planes representing the second line

$3(3r - 4) - 2(5r - 6) + (-2r + 1) + 5 = 0 \quad \Rightarrow \quad r = 2$

hence coordinate of point (2,4,–3)

and $\quad 4(2) + 3(4) - 4(-3) - 3k = 0 \Rightarrow k = \frac{32}{3}$

Hindi. $L_1 : \frac{x+4}{3} = \frac{y+6}{5} = \frac{z-1}{-2} = r \; ; \; L_2 : 3x - 2y + z + 5 = 0 = 4x + 3y - 4z - 3k$

प्रथम रेखा पर कोई बिन्दु $(3r - 4, 5r - 6, -2r + 1)$ है।

जैसाकि रेखाएँ समतलीय हैं, यह बिन्दु उन दोनों समतलों पर स्थित होना चाहिए जोकि द्वितीय रेखा को प्रदर्शित करतें हैं

$3(3r - 4) - 2(5r - 6) + (-2r + 1) + 5 = 0 \quad \Rightarrow \quad r = 2$

अतः बिन्दु के निर्देशांक (2,4,–3) है

और $\quad 4(2) + 3(4) - 4(-3) - 3k = 0 \Rightarrow k = \frac{32}{3}$

14. About the line $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{1}$ the plane $3x + 4y + 6z + 7 = 0$ is rotated till the plane passes through the origin. Now $4x + \alpha y + \beta z = 0$ is the equation of plane in new position. The value of $\alpha^2 + \beta^2$ is

रेखा $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{1}$ के सापेक्ष समतल $3x + 4y + 6z + 7 = 0$ को जबतक घुमाया जाता है तब तक यह मूल बिन्दु से नही गुजरता है अब नयी स्थिति में समतल की समीकरण $4x + \alpha y + \beta z = 0$ है तब $\alpha^2 + \beta^2$ है –

Ans. **32.00**

Sol. Since $3(2) + 4(-3) + 6(1) = 0$ and $3(1) + 4(2) + 6(-3) + 7 = 0$

$\therefore$ the line $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{1}$ lies in the plane $3x + 4y + 6z + 7 = 0$.

In the new position again the line lies in the plane. Let the equation of the new position of the plane be $ax + by + cz = 0$, then $2a - 3b + c = 0$ and $a + 2b - 3c = 0$

$\therefore \frac{a}{9-2} = \frac{b}{1+6} = \frac{c}{4+3}$ i.e. $a = b = c$

$\therefore$ equation of the required plane is $x + y + z = 0$

Hindi $\because$ $3(2) + 4(-3) + 6(1) = 0$ और $3(1) + 4(2) + 6(-3) + 7 = 0$

$\therefore$ रेखा $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{1}$ समतल $3x + 4y + 6z + 7 = 0$ पर स्थित है

पुनः नयी स्थिति में रेखा समतल पर स्थित है अतः मानाकि नयी स्थिति में, समतल का समीकरण $ax + by + cz = 0$ है, तो

$2a - 3b + c = 0$ और $a + 2b - 3c = 0$

$\therefore \frac{a}{9-2} = \frac{b}{1+6} = \frac{c}{4+3}$ अर्थात् $a = b = c$

$\therefore$ अतः अभीष्ट समतल का समीकरण $x + y + z = 0$

15. The value of $\tan^3\theta$, where θ is the acute angle between the plane faces of a regular tetrahedron, is

$\tan^3\theta$ का मान होगा जहाँ θ, समचतुष्फलक के समतल फलकों के मध्य का न्यूनकोण है– **[16JM120046]**

Ans. **22.62 or 22.63**

Sol. Since tetrahedron is regular AB = BC = AC = DC and angle between two adjcant side = $\pi/3$

consider planes ABD and DBC vector, normal to plane ABD is = $\vec{a} \times \vec{b}$

vector, normal to plane DBC is = $\vec{b} \times \vec{c}$ angle between these planes is angle between

vectors $(\vec{a} \times \vec{b})$ & $(\vec{b} \times \vec{c})$

$$\Rightarrow \cos\theta = \frac{(\vec{a} \times \vec{b}).(\vec{b} \times \vec{c})}{|\vec{a} \times \vec{b}|\,|\vec{b} \times \vec{c}|} = \frac{-\frac{1}{4}|\vec{b}|^2\,|\vec{a}|\,|\vec{c}|}{\frac{3}{4}|\vec{a}|\,|\vec{b}|^2|\vec{c}|} = -\frac{1}{3}$$

Since acute angle is required $\theta = \cos^{-1}\left(\frac{1}{3}\right)$ $\Rightarrow$ $\tan\theta = 2\sqrt{2} \Rightarrow \tan^3\theta = 16\sqrt{2}$

Hindi चूंकि दिया गया चतुष्फलक समचतुष्फलक है अतः AB = BC = AC = DC और दो आसन्न भुजाओं के मध्य कोण $\pi/3$ है।

माना ABD और DBC दो समतल हैं समतल ABD के लम्बवत् सदिश = $\vec{a} \times \vec{b}$

समतल DBC के लम्बवत् सदिश = $\vec{b} \times \vec{c}$

स्पष्टतः सदिशों $(\vec{a} \times \vec{b})$ और $(\vec{b} \times \vec{c})$ के मध्य बनने वाला कोण ही

इन समतलों के मध्य बनने वाला कोण हैं।

$$\Rightarrow \quad \cos\theta = \frac{(\vec{a}\times\vec{b}).(\vec{b}\times\vec{c})}{|\vec{a}\times\vec{b}|\ |\vec{b}\times\vec{c}|} = \frac{-\frac{1}{4}|\ \vec{b}\ |^2\ |\ \vec{a}\ |\ |\ \vec{c}\ |}{\frac{3}{4}|\ \vec{a}\ |\ \ |\ \vec{b}\ |^2|\ \vec{c}\ |} = -\frac{1}{3}$$

$\therefore$ अभीष्ट न्यून कोण $\theta = \cos^{-1}\left(\frac{1}{3}\right) \Rightarrow \quad \tan\theta = 2\sqrt{2} \Rightarrow \tan^3\theta = 16\sqrt{2}$

16. R and r are the circum–radius and in–radius of a regular tetrahedron respectively in terms of the length k of each edge. If $R^2 + r^2 = \lambda k^2$ then value of λ is

R तथा r क्रमशः समचतुष्फलक की परित्रिज्या और अन्तःत्रिज्या प्रत्येक किनारों के लम्बाई k के पदों में है। यदि $R^2 + r^2 = \lambda k^2$ तब λ का मान होगा –

Ans. **00.41 or 00.42**

Sol. circum-radius $\equiv$ distance of circum centre from any of the vertex

$\equiv$ distance of $\frac{\vec{a}+\vec{b}+\vec{c}}{4}$ from vertex $D(\vec{0})$ [tetrahedron is regular]

$$\text{Circumradius} = \frac{1}{4}|\vec{a}+\vec{b}+\vec{c}| = \frac{1}{4}\sqrt{\vec{a}^2+\vec{b}^2+\vec{c}^2+2(\vec{a}.\vec{b}+\vec{b}.\ \vec{c}+\vec{c}.\vec{a})}$$

$$= \frac{1}{4}\sqrt{k^2+k^2+k^2+2\left(\frac{k^2}{2}+\frac{k^2}{2}+\frac{k^2}{2}\right)} = \frac{1}{4}\sqrt{6k^2} = \sqrt{\frac{3}{8}}\ k$$

$$\frac{r}{R} = \frac{1}{3} \Rightarrow \quad r = \frac{R}{3} = \frac{k}{\sqrt{24}} \quad \Rightarrow \quad R = \sqrt{\frac{3}{8}}\ k \ \ \& \ r = \frac{k}{\sqrt{24}} \quad \Rightarrow \quad R^2 + r^2 = \frac{5}{12}k^2$$

$\Rightarrow$ minimum value of p + q = 17

Hindi परित्रिज्या $\equiv$ परिकेन्द्र की किसी भी शीर्ष से दूरी

$\equiv \frac{\vec{a}+\vec{b}+\vec{c}}{4}$ की शीर्ष $D(\vec{0})$ से दूरी [जैसा कि चतुष्फलक, एक समचतुष्फलक है]

$$\therefore \quad \text{परित्रिज्या} = \frac{1}{4}|\vec{a}+\vec{b}+\vec{c}| = \frac{1}{4}\sqrt{\vec{a}^2+\vec{b}^2+\vec{c}^2+2(\vec{a}.\vec{b}+\vec{b}.\ \vec{c}+\vec{c}.\vec{a})}$$

$$= \frac{1}{4}\sqrt{k^2+k^2+k^2+2\left(\frac{k^2}{2}+\frac{k^2}{2}+\frac{k^2}{2}\right)} = \frac{1}{4}\sqrt{6k^2} = \sqrt{\frac{3}{8}}\ k$$

$$\frac{r}{R} = \frac{1}{3} \Rightarrow \quad r = \frac{R}{3} = \frac{k}{\sqrt{24}} \quad \Rightarrow \quad R = \sqrt{\frac{3}{8}}\ k \ \ \& \ r = \frac{k}{\sqrt{24}} \quad \Rightarrow \quad R^2 + r^2 = \frac{5}{12}k^2$$

$\Rightarrow$ p + q = 17 का न्यूनतम मान है

17. A line L on the plane 2x + y – 3z + 5 = 0 is at a distance 3 unit from the point P(1, 2, 3). A spider starts moving from point A and after moving 4 units along the line $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{-3}$ it reaches to point P. and from P it jumps to line L along the shortest distance and then moves 12 units along the line L to reach at point B. The distance between points A and B is **[16JM120047]**

समतल 2x + y – 3z + 5 = 0 पर रेखा L, बिन्दु P(1, 2, 3) से 3 इकाई दूरी पर है। एक मकड़ी बिन्दु A से गति करती हुई रेखा $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{-3}$ के अनुदिश 4 इकाई दूरी पर बिन्दु P पर पहुँचती है तथा बिन्दु P से यह रेखा L पर लघुत्तम दूरी की रेखा के अनुदिश उछलती है। तब रेखा L के अनुदिश, 12 इकाई गति करके बिन्दु B पर पहुँचती है। बिन्दुओं A तथा B के मध्य न्यूनतम दूरी ज्ञात कीजिए।

Ans. **13.00**

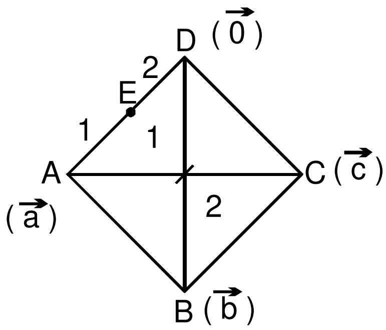
Sol. $\sqrt{3^2+4^2+12^2} = 13$

18. The length of edge of a regular tetrahedron D-ABC is 'a'. Point E & F are taken on the edges AD and BD respectively. Such that E divide $\overrightarrow{DA}$ and F divide $\overrightarrow{BD}$ in the ratio 2 : 1 each. The area of ΔCEF is equal to $\frac{\sqrt{3}}{k}a^2$, then value of k is :

समचतुष्फलक D-ABC की भुजाओं की लम्बाई 'a' है बिन्दु E और F की भुजाओं AD तथा BD पर क्रमशः है तथा E, $\overrightarrow{DA}$ और F, $\overrightarrow{BD}$ को अनुपात 2 : 1 में विभाजित करता है ΔCEF का क्षेत्रफल $\frac{\sqrt{3}}{k}a^2$ हो तो k का मान होगा –

Ans. **07.20**

Sol.



$|\vec{a}| = |\vec{b}| = |\vec{c}| = |\vec{a}-\vec{b}| = |\vec{c}-\vec{b}| = |\vec{a}-\vec{c}| = a$

On solving we get हल करने पर

$\vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{a}.\vec{c} = \frac{a^2}{2} \Rightarrow |\vec{a}| = |\vec{b}| = |\vec{c}| = a$

$E = \left(\frac{2\vec{a}}{3}\right)$ & $F = \left(\frac{\vec{b}}{3}\right)$ area of ΔCEF का क्षेत्रफल $= \frac{1}{2}|\overrightarrow{CE} \times \overrightarrow{CF}|$

$= \frac{1}{2}\left|\left(\frac{2\vec{a}}{3}-\vec{c}\right)\times\left(\frac{\vec{b}}{3}-\vec{c}\right)\right| = \frac{1}{2}\left|\frac{2}{9}(\vec{a}\times\vec{b})+\frac{2}{3}(\vec{c}\times\vec{a})+\frac{1}{3}(\vec{b}\times\vec{c})\right| = \frac{1}{2}\left|\left(\frac{2+6-3}{9}\right)\vec{a}\times\vec{b}\right|$

$= \frac{1}{2}.\frac{5}{9}|\vec{a}\times\vec{b}| \Rightarrow \frac{5}{18}.|\vec{a}\times\vec{b}| \quad \Rightarrow \frac{5}{18}.|\vec{a}|\,|\vec{b}|\,\sin 60 \Rightarrow \frac{5}{18}.a^2.\frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{36}a^2$

19. If 'd' be the shortest distance between the lines $\frac{y}{b}+\frac{z}{c} = 1; x = 0$ and $\frac{x}{a}-\frac{z}{c} = 1; y = 0$ and if $\frac{\lambda}{d^2} = \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$ then λ is

यदि रेखाओं $\frac{y}{b}+\frac{z}{c} = 1; x = 0$ तथा $\frac{x}{a}-\frac{z}{c} = 1, y = 0$ के मध्य लघुत्तम दूरी 'd' है तथा यदि $\frac{\lambda}{d^2} = \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$ तब λ का मान है –

Ans. **04.00**

Sol. $L_1 : \frac{x}{0} = \frac{y}{b} = \frac{z-c}{-c} = r$; $L_2 : \frac{x}{a} = \frac{y}{0} = \frac{z+c}{c} = \ell$

Dr's of AB are $-a\ell, br, -cr - c\ell + 2c$ $\Rightarrow$ AB is perpendicular to both the lines

$0(-a\ell) + b.\,br + (-c)(-cr - c\ell + 2c) = 0 \Rightarrow (b^2 + c^2)r + c^2\ell = 2c^2$ .....(1)

and $a(-a\ell) + 0(br) + c(-cr - c\ell + 2c) = 0 \Rightarrow -(a^2 + c^2)\ell - c^2r + 2c^2 = 0$

$(a^2 + c^2)\ell + c^2r = 2c^2$ .....(2)

from (1) & (2)

$$\ell = \frac{2b^2c^2}{a^2b^2+b^2c^2+c^2a^2}\ , \ r = \frac{2a^2c^2}{a^2b^2+b^2c^2+c^2a^2}$$

$$A\left(0\ ,\ \frac{2a^2bc^2}{a^2b^2+b^2c^2+c^2a^2},\ \ c\left(\frac{a^2b^2+b^2c^2-c^2a^2}{a^2b^2+b^2c^2+c^2a^2}\right)\right)$$

$$B\left(\frac{2ab^2c^2}{a^2b^2+b^2c^2+c^2a^2},\ \ 0\ ,\ \ c\left(\frac{b^2c^2-a^2b^2-c^2a^2}{a^2b^2+b^2c^2+c^2a^2}\right)\right)$$

$$d^2 = \frac{4a^2b^4c^4}{(a^2b^2+b^2c^2+c^2a^2)^2} + \frac{4a^4b^2c^4}{(a^2b^2+b^2c^2+c^2a^2)^2} + \frac{4c^2(a^4b^4)}{(a^2b^2+b^2c^2+c^2a^2)^2}$$

$$\frac{4}{d^2} = \frac{(a^2b^2+b^2c^2+c^2a^2)^2}{a^2b^4c^4+a^4b^2c^4+a^4b^4c^2} = \frac{a^2b^2+b^2c^2+c^2a^2}{a^2b^2c^2} \Rightarrow \frac{4}{d^2} = \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

Hindi $L_1 : \frac{x}{0} = \frac{y}{b} = \frac{z-c}{-c} = r$; $L_2 : \frac{x}{a} = \frac{y}{0} = \frac{z+c}{c} = \ell$

AB के दिक् अनुपात है $-a\ell$, br, $-cr - c\ell + 2c$

AB दोनों रेखाओं के लम्बवत् है

$0(-a\ell) + b.\ br + (-c)\ (-cr - c\ell + 2c) = 0$

$(b^2 + c^2)\ r + c^2\ell = 2c^2$ ....(1)

तथा $a(-a\ell) + 0(br) + c\ (-cr - c\ell + 2c) = 0$

$-(a^2 + c^2)\ell - c^2r + 2c^2 = 0$

$(a^2 + c^2)\ell + c^2r = 2c^2$ ....(2)

(1) और (2) से

$$\ell = \frac{2b^2c^2}{a^2b^2+b^2c^2+c^2a^2}\ , \ r = \frac{2a^2c^2}{a^2b^2+b^2c^2+c^2a^2}$$

$$A\left(0\ ,\ \frac{2a^2bc^2}{a^2b^2+b^2c^2+c^2a^2},\ \ c\left(\frac{a^2b^2+b^2c^2-c^2a^2}{a^2b^2+b^2c^2+c^2a^2}\right)\right)$$

$$B\left(\frac{2ab^2c^2}{a^2b^2+b^2c^2+c^2a^2},\ \ 0\ ,\ \ c\left(\frac{b^2c^2-a^2b^2-c^2a^2}{a^2b^2+b^2c^2+c^2a^2}\right)\right)$$

$$d^2 = \frac{4a^2b^4c^4}{(a^2b^2+b^2c^2+c^2a^2)^2} + \frac{4a^4b^2c^4}{(a^2b^2+b^2c^2+c^2a^2)^2} + \frac{4c^2(a^4b^4)}{(a^2b^2+b^2c^2+c^2a^2)^2}$$

$$\frac{4}{d^2} = \frac{(a^2b^2+b^2c^2+c^2a^2)^2}{a^2b^4c^4+a^4b^2c^4+a^4b^4c^2} = \frac{a^2b^2+b^2c^2+c^2a^2}{a^2b^2c^2} \Rightarrow \frac{4}{d^2} = \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

Exercise-2

PART - III : ONE OR MORE THAN ONE OPTIONS CORRECT TYPE

भाग - III : एक या एक से अधिक सही विकल्प प्रकार

1. A vector $\vec{a}$ has components 2p and 1 with respect to a rectangular cartesian system. The system is rotated through a certain angle about the origin in the counterclockwise sense. If with respect to the new system, $\vec{a}$ has components $p+1$ and 1, then

(A*) $p = -\frac{1}{3}$ (B*) $p = 1$ (C) $p = -1$ (D) $p = \frac{1}{3}$

कार्तीय निर्देशांक निकाय के सापेक्ष एक सदिश $\vec{a}$ के घटक क्रमशः 2p एवं 1 हैं। निकाय को मूलबिन्दु के सापेक्ष वामावर्त दिशा में एक निश्चित कोण से घुमाया जाता है। यदि नये निकाय के सापेक्ष सदिश $\vec{a}$ के घटक $p+1$ एवं 1 हैं, तो

(A*) $p = -\frac{1}{3}$ (B*) $p = 1$ (C) $p = -1$ (D) $p = \frac{1}{3}$

Sol. Before rotation $\vec{a} = 2p\hat{i} + \hat{j}$ after rotation $\vec{a} = (p+1)\hat{i}' + \hat{j}'$. Since length of vector remains unaltered

$\sqrt{4p^2+1} = \sqrt{(p+1)^2+1} \Rightarrow 4p^2 = (p+1)^2$

$\Rightarrow p + 1 = \pm 2p \qquad \Rightarrow p = 1$ or $-\frac{1}{3}$

Hindi निकाय के घूर्णन से पूर्व $\vec{a} = 2p\hat{i} + \hat{j}$. घूर्णन के बाद $\vec{a} = (p+1)\hat{i}' + \hat{j}'$. चूंकि घूर्णन के बाद भी सदिश की लम्बाई तो अपरिवर्तित ही रहती है

$\sqrt{4p^2+1} = \sqrt{(p+1)^2+1} \Rightarrow 4p^2 = (p+1)^2$

$\Rightarrow p + 1 = \pm 2p \qquad \Rightarrow p = 1$ or $-\frac{1}{3}$

2. If $\vec{z}_1 = a\hat{i} + b\hat{j}$ and $\vec{z}_2 = c\hat{i} + d\hat{j}$ are two vectors in $\hat{i}$ and $\hat{j}$ system, where $|\vec{z}_1| = |\vec{z}_2| = r$ and $\vec{z}_1 \cdot \vec{z}_2 = 0$, then $\vec{w}_1 = a\hat{i} + c\hat{j}$ and $\vec{w}_2 = b\hat{i} + d\hat{j}$ satisfy : **[16JM120048]**

(A*) $|\vec{w}_1| = r$ (B*) $|\vec{w}_2| = r$ (C*) $\vec{w}_1 . \vec{w}_2 = 0$ (D) $|\vec{w}_1| \neq |\vec{w}_2|$

यदि $\hat{i}$ तथा $\hat{j}$ निकाय में दो सदिश $\vec{z}_1 = a\hat{i} + b\hat{j}$ और $\vec{w}_2 = b\hat{i} + d\hat{j}$ हैं, जहाँ $|\vec{z}_1| = |\vec{z}_2| = r$ और $\vec{z}_1 \cdot \vec{z}_2 = 0$. तो $\vec{w}_1 = a\hat{i} + c\hat{j}$ और $\vec{w}_2 = b\hat{i} + d\hat{j}$ संतुष्ट करते हैं –

(A*) $|\vec{w}_1| = r$ (B*) $|\vec{w}_2| = r$ (C*) $\vec{w}_1 . \vec{w}_2 = 0$ (D) $|\vec{w}_1| \neq |\vec{w}_2|$

Sol. As given $a^2 + b^2 = c^2 + d^2$, $ac + bd = 0$ from second relation $ac = -bd$ $\quad \Rightarrow \frac{a}{b} = -\frac{d}{c}$

using in first relation $\lambda^2 b^2 + b^2 = c^2 + \lambda^2 c^2$

$\Rightarrow \quad b^2 = c^2 \;\; \& \;\; a^2 = d^2 \quad \Rightarrow \quad (b = c, a = -d)$ or $(b = -c, \; a = d)$.

Now $|\vec{w}_1| = \sqrt{a^2+c^2} = \sqrt{a^2+b^2}$ $|\vec{w}_2| = \sqrt{b^2+d^2} = \sqrt{c^2+d^2}$

$\vec{w}_1.\vec{w}_2 = ab + cd = ab + b(-a) = 0$

Hindi जैसा कि दिया है $a^2 + b^2 = c^2 + d^2$, $ac + bd = 0$ द्वितीय सम्बन्ध से $ac = -bd$ $\quad \Rightarrow \frac{a}{b} = -\frac{d}{c}$

प्रथम सम्बन्ध में उपयोग करने पर $\lambda^2 b^2 + b^2 = c^2 + \lambda^2 c^2$

$\Rightarrow \quad b^2 = c^2 \;\; \& \;\; a^2 = d^2 \quad \Rightarrow \quad (b = c, a = -d)$ or $(b = -c, \; a = d)$

अब $|\vec{w}_1| = \sqrt{a^2+c^2} = \sqrt{a^2+b^2}$, $|\vec{w}_2| = \sqrt{b^2+d^2} = \sqrt{c^2+d^2}$

$\bar{w}_1.\bar{w}_2 = ab + cd = ab + b(-a) = 0$

3. If a, b, c, x, y, z ∈ R such that ax + by + cz = 2, then which of the following is always true

यदि a, b, c, x, y, z ∈ R इस प्रकार है कि ax + by + cz = 2 तो निम्न में से कौनसे सदैव सत्य है

(A*) $(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) \geq 4$ (B*) $(x^2 + b^2 + z^2)(a^2 + y^2 + c^2) \geq 4$

(C*) $(a^2 + y^2 + z^2)(x^2 + b^2 + c^2) \geq 4$ (D*) $(a^2 + b^2 + z^2)(x^2 + y^2 + c^2) \geq 4$

Sol. Let $\vec{A} = a\hat{i} + b\hat{j} + c\hat{k}$ & $\vec{B} = x\hat{i} + y\hat{j} + z\hat{k}$

given that $\vec{A}.\vec{B} = 2 \Rightarrow |\vec{A}|\,|\vec{B}|\cos\theta = 2$

$\Rightarrow \sqrt{x^2+y^2+z^2}\,.\sqrt{a^2+b^2+c^2} \geq 2$

$\Rightarrow (a^2 + b^2 + c^2)(x^2 + y^2 + z^2) \geq 4$

Similarly the others.

माना $\vec{A} = a\hat{i} + b\hat{j} + c\hat{k}$ एवं $\vec{B} = x\hat{i} + y\hat{j} + z\hat{k}$

दिया गया है $\vec{A}.\vec{B} = 2$

$\Rightarrow |\vec{A}|\,|\vec{B}|\cos\theta = 2 \Rightarrow \sqrt{x^2+y^2+z^2}\,.\sqrt{a^2+b^2+c^2} \geq 2$

$\Rightarrow (a^2 + b^2 + c^2)(x^2 + y^2 + z^2) \geq 4$

इसी प्रकार अन्य

4. The direction cosines of the lines bisecting the angle between the lines whose direction cosines are ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 and the angle between these lines is θ, are **[16JM120049]**

उन रेखाओं की दिक् कोज्याऐं जो ℓ_1, m_1, n_1 और ℓ_2, m_2, n_2 दिक्कोज्याओं वाली रेखाओं के मध्य कोण को समद्विभाजित करती हैं जबकि इन रेखाओं के मध्य कोण θ है, होंगी –

(A) $\dfrac{\ell_1-\ell_2}{\cos\frac{\theta}{2}}, \dfrac{m_1-m_2}{\cos\frac{\theta}{2}}, \dfrac{n_1-n_2}{\cos\frac{\theta}{2}}$ (B*) $\dfrac{\ell_1+\ell_2}{2\cos\frac{\theta}{2}}, \dfrac{m_1+m_2}{2\cos\frac{\theta}{2}}, \dfrac{n_1+n_2}{2\cos\frac{\theta}{2}}$

(C*) $\dfrac{\ell_1+\ell_2}{2\sin\frac{\theta}{2}}, \dfrac{m_1+m_2}{2\sin\frac{\theta}{2}}, \dfrac{n_1+n_2}{2\sin\frac{\theta}{2}}$ (D*) $\dfrac{\ell_1-\ell_2}{2\sin\frac{\theta}{2}}, \dfrac{m_1-m_2}{2\sin\frac{\theta}{2}}, \dfrac{n_1-n_2}{2\sin\frac{\theta}{2}}$

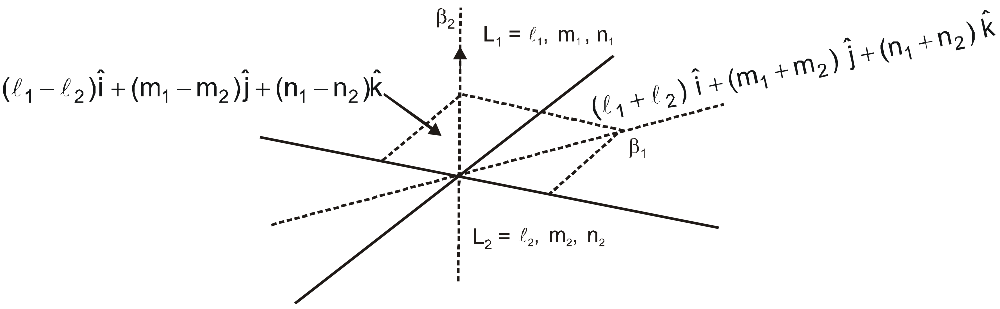
Sol. $\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$; Dc's of β_1 (bisector)

$$\frac{\ell_1+\ell_2}{\sqrt{(\ell_1+\ell_2)^2+(m_1+m_2)^2+(n_1+n_2)^2}} = \frac{\ell_1+\ell_2}{\sqrt{2+2(\ell_1\ell_2+m_1m_2+n_1n_2)}} = \frac{\ell_1+\ell_2}{\sqrt{2+2\cos\theta}} = \frac{\ell_1+\ell_2}{2\cos\theta/2}$$

Similarly $\dfrac{m_1+m_2}{2\cos\theta/2}, \dfrac{n_1+n_2}{2\cos\theta/2}$

Similarly Dc's for bisector β_2

$$\frac{\ell_1-\ell_2}{2\sin\frac{\theta}{2}}, \frac{m_1-m_2}{2\sin\frac{\theta}{2}}, \frac{n_1-n_2}{2\sin\frac{\theta}{2}}$$



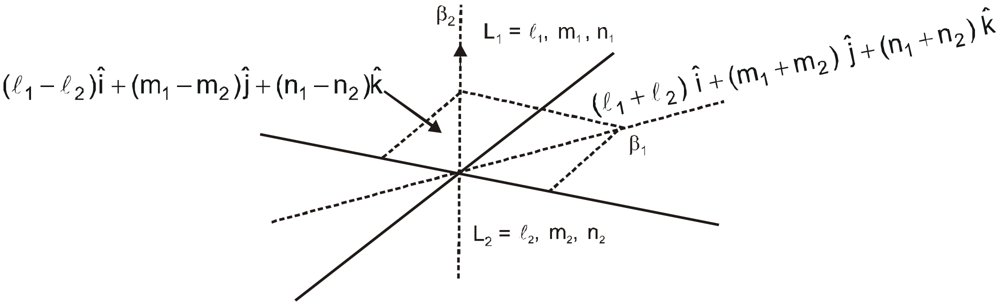
Hindi $\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$. β_1 (अर्धक) की दिक्कोज्याएँ हैं

$$\frac{\ell_1+\ell_2}{\sqrt{(\ell_1+\ell_2)^2+(m_1+m_2)^2+(n_1+n_2)^2}} = \frac{\ell_1+\ell_2}{\sqrt{2+2(\ell_1\ell_2+m_1m_2+n_1n_2)}} = \frac{\ell_1+\ell_2}{\sqrt{2+2\cos\theta}} = \frac{\ell_1+\ell_2}{2\cos\theta/2}$$

इसी प्रकार $\qquad \dfrac{m_1+m_2}{2\cos\theta/2}, \dfrac{n_1+n_2}{2\cos\theta/2}$

इसी प्रकार अर्द्धक β_2 की दिक्कोज्याएँ हैं

$$\frac{\ell_1-\ell_2}{2\sin\frac{\theta}{2}}, \frac{m_1-m_2}{2\sin\frac{\theta}{2}}, \frac{n_1-n_2}{2\sin\frac{\theta}{2}}$$



5. The value(s) of $\alpha \in [0, 2\pi]$ for which vector $\vec{a} = \hat{i}+3\hat{j}+(\sin 2\alpha)\ \hat{k}$ makes an obtuse angle with the z-axis and the vectors $\vec{b} = (\tan\alpha)\ \hat{i}-\hat{j}+2\sqrt{\sin\frac{\alpha}{2}}\ \hat{k}$ and $\vec{c} = (\tan\alpha)\,\hat{i} + (\tan\alpha)\ \hat{j} - 3\sqrt{\text{cosec}\frac{\alpha}{2}}\ \hat{k}$ are orthogonal, is/are

$\alpha \in [0, 2\pi]$ के (का) वह मान जिनके लिए सदिश $\vec{a} = \hat{i}+3\hat{j}+(\sin 2\alpha)\ \hat{k}$, z-अक्ष के साथ अधिक कोण बनाता है और सदिश $\vec{b} = (\tan\alpha)\ \hat{i}-\hat{j}+2\sqrt{\sin\frac{\alpha}{2}}\ \hat{k}$ एवं $\vec{c} = (\tan\alpha)\hat{i} + (\tan\alpha)\hat{j} - 3\sqrt{\text{cosec}\frac{\alpha}{2}}\ \hat{k}$ परस्पर लम्बवत् है, होगा–

(A) $\tan^{-1} 3$ (B*) $\pi - \tan^{-1} 2$ (C) $\pi + \tan^{-1} 3$ (D*) $2\pi - \tan^{-1} 2$

Sol. Since $\vec{a}$ makes obtuse angle with z-axis

$\therefore \quad \dfrac{\sin 2\alpha}{\sqrt{1+9+\sin^2 2\alpha}} < 0 \qquad$ i.e. $\quad \sin 2\alpha < 0$

$\therefore \quad$ either $\dfrac{\pi}{2} < \alpha < \pi$ or $\dfrac{3\pi}{2} < \alpha < 2\pi \qquad$ ......(i)

since $\vec{b}$ and $\vec{c}$ are orthogonal

$\therefore \quad \tan^2\alpha - \tan\alpha - 6 = 0 \qquad$ i.e. $\quad \tan\alpha = 3, -2 \qquad$ ......(ii)

from (i) and (ii), we get

$\tan\alpha = -2 \qquad \therefore \quad \alpha = \pi - \tan^{-1} 2$ or $\alpha = 2\pi - \tan^{-1} 2$

Hindi चूंकि $\vec{a}$, z-अक्ष के साथ अधिक कोण बनाता है।

$\therefore \quad \dfrac{\sin 2\alpha}{\sqrt{1+9+\sin^2 2\alpha}} < 0 \qquad$ अर्थात् $\quad \sin 2\alpha < 0$

$\therefore \quad$ या तो $\dfrac{\pi}{2} < \alpha < \pi$ या $\dfrac{3\pi}{2} < \alpha < 2\pi \qquad$ ......(i)

चूंकि $\vec{b}$ और $\vec{c}$ लाम्बिक हैं।

$\therefore \quad \tan^2\alpha - \tan\alpha - 6 = 0 \qquad$ अर्थात् $\quad \tan\alpha = 3, -2 \qquad$ ......(ii)

समीकरण (i) और (ii) से $\quad \tan\alpha = -2 \qquad \therefore \quad \alpha = \pi - \tan^{-1} 2$ या $\alpha = 2\pi - \tan^{-1} 2$

6. The vector $\hat{i}+x\hat{j}+3\hat{k}$ is rotated through an angle of $\cos^{-1}\dfrac{11}{14}$ and doubled in magnitude, then it becomes $4\hat{i}+(4x-2)\hat{j}+2\hat{k}$. The value of 'x' **CANNOT** be : **[16JM120050]**

एक सदिश $\hat{i} + x\hat{j} + 3\hat{k}$ को कोण $\cos^{-1}\dfrac{11}{14}$ से घुमाया जाता है, तथा फिर इसका परिमाण दुगना कर दिया जाता है, तो यह सदिश $4\hat{i} + (4x-2)\hat{j} + 2\hat{k}$ बन जाता है। 'x' का मान नही हो सकता है–

(A*) $-\dfrac{2}{3}$ (B*) $\dfrac{2}{3}$ (C*) $-\dfrac{20}{17}$ (D) 2

Sol. $|\overrightarrow{AC}|^2 = |2\overrightarrow{AB}|^2 \Rightarrow |4\hat{i} + (4x-2)\hat{j} + 2\hat{k}|^2 = 4|(\hat{i} + x\hat{j} + 3\hat{k})|^2$

$\Rightarrow 16 + (4x-2)^2 + 4 = 4(1 + x^2 + 9) \Rightarrow 20 + 16x^2 + 4 - 16x = 4 + 4x^2 + 36$

$\Rightarrow 12x^2 - 16x - 16 = 0 \Rightarrow 3x^2 - 4x - 4 = 0$

$\Rightarrow x = 2, -\dfrac{2}{3}$ ...(i)

angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is $\cos\theta = \dfrac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}|\,|\overrightarrow{AC}|} = \dfrac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{2\,|\overrightarrow{AB}|^2}$

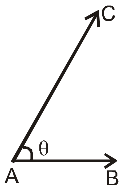
$\Rightarrow \dfrac{11}{14} = \dfrac{(\hat{i} + x\hat{j} + 3\hat{k})\cdot(4\hat{i} + (4x-2)\hat{j} + 2\hat{k})}{2(1+x^2+9)} = \dfrac{4 + x(4x-2) + 6}{2x^2 + 20}$

$\Rightarrow 11x^2 + 110 = 70 + 28x^2 - 14x$

$\Rightarrow 17x^2 - 14x - 40 = 0 \quad \therefore \quad x = 2, -\dfrac{20}{17}$ ....(ii)

from (i) and (ii)

$x = 2$



Hindi $|\overrightarrow{AC}|^2 = |2\overrightarrow{AB}|^2 \Rightarrow |4\hat{i} + (4x-2)\hat{j} + 2\hat{k}|^2 = 4|(\hat{i} + x\hat{j} + 3\hat{k})|^2$

$\Rightarrow 16 + (4x-2)^2 + 4 = 4(1 + x^2 + 9) \Rightarrow 20 + 16x^2 + 4 - 16x = 4 + 4x^2 + 36$

$\Rightarrow 12x^2 - 16x - 16 = 0 \Rightarrow 3x^2 - 4x - 4 = 0$

$\Rightarrow x = 2, -\dfrac{2}{3}$ ....(i)

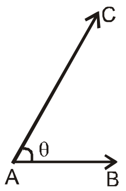
$\overrightarrow{AB}$ और $\overrightarrow{AC}$ के बीच कोण $\cos\theta = \dfrac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}|\,|\overrightarrow{AC}|} = \dfrac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{2\,|\overrightarrow{AB}|^2}$

$\Rightarrow \dfrac{11}{14} = \dfrac{(\hat{i} + x\hat{j} + 3\hat{k})\cdot(4\hat{i} + (4x-2)\hat{j} + 2\hat{k})}{2(1+x^2+9)} = \dfrac{4 + x(4x-2) + 6}{2x^2 + 20}$

$\Rightarrow 11x^2 + 110 = 70 + 28x^2 - 14x$

$\Rightarrow 17x^2 - 14x - 40 = 0 \quad \therefore \quad x = 2, -\dfrac{20}{17}$ ......(ii)

समीकरण (i) और (ii) से $x = 2$

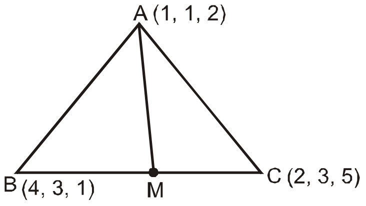


7. The vertices of a triangle are A (1, 1, 2), B(4, 3, 1) and C(2, 3, 5). A vector representing the bisector of the angle A is :

एक त्रिभुज के शीर्ष A (1, 1, 2), B(4, 3, 1) और C(2, 3, 5) हैं। कोण A के समद्विभाजक को निरूपित करने वाला सदिश है –

(A*) $2\hat{i}-4\hat{k}$ (B*) $-2\hat{i}+4\hat{k}$ (C*) $-2\hat{i}-2\hat{j}-\hat{k}$ (D*) $2\hat{i}+2\hat{j}+\hat{k}$

Sol. AB = $\sqrt{9+4+1}=\sqrt{14}$; AC = $\sqrt{1+4+9}=\sqrt{14}$



$M \equiv (3, 3, 3)$; $\overrightarrow{AM} = 2\hat{i}+2\hat{j}+\hat{k}$

8. The vector $\vec{c}$, parallel to the internal bisector of the angle between the vectors $\vec{a}=7\hat{i}-4\hat{j}-4\hat{k}$ and $\vec{b}=-2\hat{i}-\hat{j}+2\hat{k}$ with $|\vec{c}| = 5\sqrt{6}$, is :

एक सदिश $\vec{c}$ की दिशा सदिशों $\vec{a}=7\hat{i}-4\hat{j}-4\hat{k}$ और $\vec{b}=-2\hat{i}-\hat{j}+2\hat{k}$ के अन्तः कोण समद्विभाजक के अनुदिश है यदि $|\vec{c}|=5\sqrt{6}$ है तो है–

(A*) $\frac{5}{3}\left(\hat{i}-7\hat{j}+2\hat{k}\right)$ (B) $\frac{5}{3}\left(\hat{i}+7\hat{j}-2\hat{k}\right)$ (C*) $\frac{5}{3}\left(-\hat{i}+7\hat{j}-2\hat{k}\right)$ (D) $\frac{5}{3}\left(-\hat{i}-7\hat{j}+2\hat{k}\right)$

Sol. any such vector $\vec{c}=\lambda\ (\hat{a}+\hat{b}) = \lambda\left(\frac{7\hat{i}-4\hat{j}-4\hat{k}}{9}+\frac{-2\hat{i}-\hat{j}+2\hat{k}}{3}\right) = \frac{\lambda}{9}\left[7\hat{i}-4\hat{j}-4\hat{k}+3(-2\hat{i}-\hat{j}+2\hat{k})\right]$

$= \frac{\lambda}{9}\left[\hat{i}-7\hat{j}+2\hat{k}\right] \Rightarrow |\vec{c}|=5\sqrt{6} \Rightarrow \left|\frac{\lambda}{9}\sqrt{1+49+4}\right| = 5\sqrt{6}$

$\Rightarrow \left|\frac{\lambda}{9}\sqrt{54}\right| = 5\sqrt{6} \Rightarrow \lambda = \pm\frac{9\times5\sqrt{6}}{\sqrt{54}} = \pm 15 \Rightarrow \vec{c}=\pm\frac{15}{9}(\hat{i}-7\hat{j}+2\hat{k}) = \pm\frac{5}{3}(\hat{i}-7\hat{j}+2\hat{k})$

Hindi ऐसे सदिश को निम्न रूप में लिखा जा सकता है–

$\vec{c}=\lambda\ (\hat{a}+\hat{b}) = \lambda\left(\frac{7\hat{i}-4\hat{j}-4\hat{k}}{9}+\frac{-2\hat{i}-\hat{j}+2\hat{k}}{3}\right) = \frac{\lambda}{9}\left[7\hat{i}-4\hat{j}-4\hat{k}+3(-2\hat{i}-\hat{j}+2\hat{k})\right]$

$= \frac{\lambda}{9}\left[\hat{i}-7\hat{j}+2\hat{k}\right] \Rightarrow |\vec{c}|=5\sqrt{6} \Rightarrow \left|\frac{\lambda}{9}\sqrt{1+49+4}\right| = 5\sqrt{6}$

$\Rightarrow \left|\frac{\lambda}{9}\sqrt{54}\right| = 5\sqrt{6} \Rightarrow \lambda = \pm\frac{9\times5\sqrt{6}}{\sqrt{54}} = \pm 15 \Rightarrow \vec{c}=\pm\frac{15}{9}(\hat{i}-7\hat{j}+2\hat{k}) = \pm\frac{5}{3}(\hat{i}-7\hat{j}+2\hat{k})$

9. A line passes through a point A with position vector $3\hat{i}+\hat{j}-\hat{k}$ and is parallel to the vector $2\hat{i}-\hat{j}+2\hat{k}$. If P is a point on this line such that AP = 15 units, then the position vector of the point P is/are **[16JM120051]**

एक रेखा बिन्दु A से गुजरती है जिसका स्थिति सदिश $3\hat{i}+\hat{j}-\hat{k}$ है और सदिश $2\hat{i}-\hat{j}+2\hat{k}$ के समान्तर है। यदि इस रेखा पर एक बिन्दु P इस प्रकार है कि AP = 15 इकाई, तो बिन्दु P का स्थिति सदिश है/हैं–

(A) $13\hat{i}+4\hat{j}-9\hat{k}$ (B*) $13\hat{i}-4\hat{j}+9\hat{k}$ (C) $7\hat{i}-6\hat{j}+11\hat{k}$ (D*) $-7\hat{i}+6\hat{j}-11\hat{k}$

Sol. line is $\vec{r}=(3\hat{i}+\hat{j}-\hat{k})+\lambda(2\hat{i}-\hat{j}+2\hat{k})$. Let position vector of point $\vec{p}$ is $\Rightarrow \vec{p} = (3\hat{i}+\hat{j}-\hat{k})+\lambda(2\hat{i}-\hat{j}+2\hat{k})$

given that $|\vec{p}-\vec{a}|=15 \Rightarrow |\vec{p}-(3\hat{i}+\hat{j}-\hat{k})| = 15$ or $|\lambda(2\hat{i}-\hat{j}+2\hat{k})|=15$

$\Rightarrow |3\lambda| = 15 \Rightarrow \lambda = \pm 5 \Rightarrow \vec{p}=(3\hat{i}+\hat{j}-\hat{k})\pm5(2\hat{i}-\hat{j}+2\hat{k})$

Hindi रेखा $\vec{r} = (3\hat{i}+\hat{j}-\hat{k}) + \lambda(2\hat{i}-\hat{j}+2\hat{k})$ है। माना बिन्दु का स्थिति सदिश $\vec{p}$ है

$\Rightarrow \quad \vec{p} = (3\hat{i}+\hat{j}-\hat{k}) + \lambda(2\hat{i}-\hat{j}+2\hat{k})$ दिया है कि $|\vec{p}-\vec{a}| = 15 \quad \Rightarrow$

$|\vec{p}-(3\hat{i}+\hat{j}-\hat{k})| = 15$ or $|\lambda(2\hat{i}-\hat{j}+2\hat{k})| = 15 \quad \Rightarrow \quad |3\lambda| = 15 \Rightarrow \lambda = \pm 5$

$\Rightarrow \vec{p} = (3\hat{i}+\hat{j}-\hat{k}) \pm 5(2\hat{i}-\hat{j}+2\hat{k})$

10. Acute angle between the lines $\frac{x-1}{\ell} = \frac{y+1}{m} = \frac{z}{n}$ and $\frac{x+1}{m} = \frac{y-3}{n} = \frac{z-1}{\ell}$ where $\ell > m > n$, and ℓ, m, n are the roots of the cubic equation $x^3 + x^2 - 4x = 4$ is equal to :

रेखाओं $\frac{x-1}{\ell} = \frac{y+1}{m} = \frac{z}{n}$ तथा $\frac{x+1}{m} = \frac{y-3}{n} = \frac{z-1}{\ell}$ के मध्य न्यूनकोण ज्ञात कीजिये जहां $\ell > m > n$, तथा ℓ, m, n त्रिघातीय समीकरण $x^3 + x^2 - 4x = 4$ के मूल हैं।

(A) $\cos^{-1}\frac{3}{\sqrt{13}}$ (B*) $\sin^{-1}\frac{\sqrt{65}}{9}$ (C*) $2\cos^{-1}\sqrt{\frac{13}{18}}$ (D) $\tan^{-1}\frac{2}{3}$

Sol. $\text{Cos}\theta = \frac{\ell m + mn + n\ell}{\ell^2 + m^2 + n^2}$ ....(i)

$x^3 + x^2 - 4x - 4 = 0 \quad \Rightarrow \quad \ell + m + n = -1 \quad \Rightarrow \quad \ell m + mn + n\ell = -4$

$(\ell + m + n)^2 = \ell^2 + m^2 + n^2 + 2(-4) \quad \Rightarrow \quad \ell^2 + m^2 + n^2 = 1 + 8 = 9$

$\therefore \quad \cos\theta = -\frac{4}{9} \quad \therefore$ acute angle between the lines is $\cos^{-1}\frac{4}{9}$

Hindi. $\text{Cos}\theta = \frac{\ell m + mn + n\ell}{\ell^2 + m^2 + n^2}$ ....(i)

$x^3 + x^2 - 4x - 4 = 0 \quad \Rightarrow \quad \ell + m + n = -1 \quad \Rightarrow \quad \ell m + mn + n\ell = -4$

$(\ell + m + n)^2 = \ell^2 + m^2 + n^2 + 2(-4) \quad \Rightarrow \quad \ell^2 + m^2 + n^2 = 1 + 8 = 9$

$\therefore \quad \cos\theta = -\frac{4}{9} \quad \therefore$ रेखाओं के मध्य न्यूनकोण $\cos^{-1}\frac{4}{9}$ है।

11. The line $\frac{x-2}{3} = \frac{y+1}{2} = \frac{z-1}{-1}$ intersects the curve $xy = c^2$, $z = 0$ if c is equal to : **[16JM120053]**

रेखा $\frac{x-2}{3} = \frac{y+1}{2} = \frac{z-1}{-1}$, वक्र $xy = c^2$, $z = 0$ को प्रतिच्छेद करती है तो c का मान है–

(A) –1 (B*) $-\sqrt{5}$ (C*) $\sqrt{5}$ (D) 1

Sol. Any pt. on line is $(3\lambda + 2, 2\lambda - 1, 1 - \lambda)$

but it lies on the curve $xy = c^2$ & $z = 0 \quad \Rightarrow \quad (3\lambda + 2)(2\lambda - 1) = c^2$ & $1 - \lambda = 0$

$\Rightarrow \quad (3\lambda + 2)(2\lambda - 1) = c^2$ & $\lambda = 1 \quad \Rightarrow \quad c^2 = 5 \quad \Rightarrow \quad c = \pm\sqrt{5}$

Hindi रेखा पर कोई बिन्दु $(3\lambda + 2, 2\lambda - 1, 1 - \lambda)$ है।

परन्तु यह वक्र $xy = c^2$ व $z = 0$ पर स्थित है $\Rightarrow \quad (3\lambda + 2)(2\lambda - 1) = c^2$ व $1 - \lambda = 0$

$\Rightarrow \quad (3\lambda + 2)(2\lambda - 1) = c^2$ व $\lambda = 1 \quad \Rightarrow \quad c^2 = 5 \quad \Rightarrow \quad c = \pm\sqrt{5}$

12. Three distinct lines $\frac{x-1}{3} = \frac{y-2}{2} = \frac{z-3}{1}$, $\frac{x-1}{5} = \frac{2y-4}{3} = \frac{3z-9}{1}$, $\frac{x-\lambda^2}{3} = \frac{y-2}{2} = \frac{z-3}{\lambda}$ are concurrent the value of λ may be :

तीन भिन्न रेखाएँ $\frac{x-1}{3}=\frac{y-2}{2}=\frac{z-3}{1}, \frac{x-1}{5}=\frac{2y-4}{3}=\frac{3z-9}{1}, \frac{x-\lambda^2}{3}=\frac{y-2}{2}=\frac{z-3}{\lambda}$ संगामी है, तो λ का मान हो सकता है

(A) 1 (B*) –1 (C) 2 (D) –2

Sol. First two lines pass through the point (1, 2, 3) the third line must pass through the same point therefor $\lambda^2 = 1 \Rightarrow \lambda = \pm 1$. But λ cannot be 1 as the third line consids with the first.

प्रथम दो रेखाएँ बिन्दु (1, 2, 3) से गुजरती है। तीसरी रेखा भी इसी बिन्दु से गुजरती है इसलिए $\lambda^2 = 1 \Rightarrow \lambda = \pm 1$ परन्तु λ का मान 1 नहीं होगा क्योंकि प्रथम, तीसरी रेखा पर संपाती है।

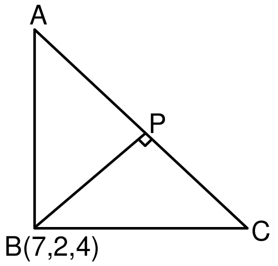
13. The line $\frac{x+6}{5}=\frac{y+10}{3}=\frac{z+14}{8}$ is the hypotenuse of an isosceles right angle triangle whose opposite vertex is (7, 2, 4) Then the equation of remaining sides is/are -

रेखा $\frac{x+6}{5}=\frac{y+10}{3}=\frac{z+14}{8}$ समद्विबाहु समकोण त्रिभुज का कर्ण है जिसका विपरीत शीर्ष (7, 2, 4) है। तब शेष भुजाओं के समीकरण होगें –

(A*) $\frac{x-7}{3}=\frac{y-2}{6}=\frac{z-4}{2}$ **(B*)** $\frac{x-7}{2}=\frac{y-2}{-3}=\frac{z-4}{6}$

(C) $\frac{x+7}{3}=\frac{y+2}{6}=\frac{z+4}{2}$ **(D)** $\frac{x+7}{2}=\frac{y+2}{-3}=\frac{z+4}{6}$

Sol.



P = (5k – 6, 3k – 10, 8k – 14)

Now अब $\overrightarrow{BP}\cdot\overrightarrow{AC} = 0$

5(5k – 13) + 3 (3k – 12) + 8 (8k – 18) = 0

$k = \frac{245}{98} = \frac{5}{2}$

$P = \left(\frac{13}{2}, -\frac{5}{2}, 6\right)$, Now अब length BP की लम्बाई $= \frac{\sqrt{98}}{2}$

Now अब $A, C = \left(\frac{13}{2} \pm \frac{5}{\sqrt{98}}\cdot\frac{\sqrt{98}}{2}, -\frac{5}{2} \pm \frac{3}{\sqrt{98}}\cdot\frac{\sqrt{98}}{2}, 6 \pm \frac{8}{\sqrt{98}}\cdot\frac{\sqrt{98}}{2}\right)$

A = (9, –1, 10)
C = (4, –4, 2)

Equation of BC का समीकरण $\frac{x-7}{3}=\frac{y-2}{6}=\frac{z-4}{2}$

Equation of AC का समीकरण $\frac{x-7}{2}=\frac{y-2}{-3}=\frac{z-4}{6}$

14. Two lines are दो रेखाऐं है

$L_1 : \frac{x-1}{1}=\frac{y-1}{2}=\frac{z-1}{2}$; $L_2 : \frac{x-1}{2}=\frac{y-1}{1}=\frac{z-1}{2}$

Equation of line passing through (2, 1, 3) and equally inclined to L_1 & L_2 is/are

बिन्दु (2, 1, 3) से गुजरने वाली रेखा का समीकरण जो L_1 तथा L_2 से बराबर झुकी हुई है

(A*) $\frac{x-2}{2}=\frac{y-1}{2}=\frac{z-3}{-3}$ (B*) $\frac{x-3}{1}=\frac{y-2}{1}=\frac{z-5}{2}$

(C*) $\frac{x-2}{1}=\frac{y-1}{1}=\frac{z-3}{3}$ (D*) $\frac{x}{2}=\frac{y+1}{2}=\frac{z-6}{-3}$

Sol. Any line which is perpendicular to direction1 + 2, 2 + 1, 2 + 2 or 1 – 2, 2 – 1, 2 – 2 i.e., 3, 3, 4 or –1, 1, 0 is always equally inclined to L_1 & L_2.

कोई रेखा जो 1 + 2, 2 + 1, 2 + 2 या 1 – 2, 2 – 1, 2 – 2 अर्थात् 3, 3, 4 या –1, 1, 0 के लम्बवत् तथा L_1 तथा L_2 के साथ बराबर झुकी हुई है।

15. Which of the followings is/are correct : **[16JM120052]**

निम्न में से कौनसा सही है।

(A*) The angle between the two straight lines $\vec{r}=3\hat{i}-2\hat{j}+4\hat{k}+\lambda(-2\hat{i}+\hat{j}+2\hat{k})$ and

$\vec{r}=\hat{i}+3\hat{j}-2\hat{k}+\mu(3\hat{i}-2\hat{j}+6\hat{k})$ is $\cos^{-1}\left(\frac{4}{21}\right)$

दो सरल रेखाओं $\vec{r}=3\hat{i}-2\hat{j}+4\hat{k}+\lambda(-2\hat{i}+\hat{j}+2\hat{k})$ और

$\vec{r}=\hat{i}+3\hat{j}-2\hat{k}+\mu(3\hat{i}-2\hat{j}+6\hat{k})$ के बीच कोण $\cos^{-1}\left(\frac{4}{21}\right)$ हैं।

(B*) $(\vec{r}.\hat{i})\ (\hat{i}\times\vec{r})+(\vec{r}.\hat{j})\ (\hat{j}\times\vec{r})+(\vec{r}.\hat{k})\ (\hat{k}\times\vec{r})=\vec{0}$.

(C*) The force determined by the vector $\vec{r}$ = (1, –8, –7) is resolved along three mutually perpendicular directions, one of which is in the direction of the vector $\vec{a}=2\hat{i}+2\hat{j}+\hat{k}$. Then the vector component of the force $\vec{r}$ in the direction of the vector $\vec{a}$ is $-\frac{7}{3}(2\hat{i}+2\hat{j}+\hat{k})$

सदिश $\vec{r}$ = (1, –8, –7) द्वारा निरूपित बल को तीन परस्पर लम्बवत् दिशाओं में वियोजित किया जाता है जिनमें से एक, सदिश $\vec{a}=2\hat{i}+2\hat{j}+\hat{k}$ की दिशा में है। तब सदिश $\vec{a}$ की दिशा में बल $\vec{r}$ का सदिश घटक $-\frac{7}{3}(2\hat{i}+2\hat{j}+\hat{k})$ है।

(D*) The cosine of the acute angle between any two diagonals of a cube is $\frac{1}{3}$.

एक घन के किन्हीं दो विकर्णो के न्यून कोण का कोज्या (cosine) $\frac{1}{3}$ है.

Sol. S_1 : Angle between the given lines is = angle between the vectors $(-2\hat{i}+\hat{j}+2\hat{k})$ and $(3\hat{i}-2\hat{j}+6\hat{k})$

दी गई रेखाओं के बीच कोण = सदिशों $(-2\hat{i}+\hat{j}+2\hat{k})$ और $(3\hat{i}-2\hat{j}+6\hat{k})$ के बीच का कोण

$$\Rightarrow \quad \cos\theta=\frac{(-2\hat{i}+\hat{j}+2\hat{k}).(3\hat{i}-2\hat{j}+6\hat{k})}{|-\ 2\hat{i}+\hat{j}+2\hat{k}|\ |3\hat{i}-2\hat{j}+6\hat{k}|}=\cos^{-1}\left(\frac{4}{21}\right)$$

S_2 : Let (मानाकि) ($\vec{r}=x\hat{i}+y\hat{j}+2\hat{k}$) $\vec{r}=x\hat{i}+y\hat{j}+2\hat{k}$

given expression is $x[y\hat{k}-z\hat{j}]+y[z\hat{i}-x\hat{k}]+z[x\hat{j}-y\hat{i}]=\vec{0}$

दिया गया व्यंजक $x[y\hat{k}-z\hat{j}]+y[z\hat{i}-x\hat{k}]+z[x\hat{j}-y\hat{i}]=\vec{0}$ है।

S_3 : Component of $\vec{r}$ in direction of $\vec{a}=\frac{\vec{r}.\vec{a}}{|\vec{a}|}\hat{a}$

$$=\left\{\frac{(\hat{i}-8\hat{j}-7\hat{k})\ .(2\hat{i}+2\hat{j}+\hat{k})}{3}\right\}\frac{(2\hat{i}+2\hat{j}+\hat{k})}{3}=-\frac{21}{9}(2\hat{i}+2\hat{j}+\hat{k})=-\frac{7}{3}(2\hat{i}+2\hat{j}+\hat{k})$$

सदिश $\vec{a}$ की दिशा में सदिश $\vec{r}$ का घटक $= \dfrac{\vec{r}.\vec{a}}{|\vec{a}|}\hat{a}$

$$= \left\{\frac{(\hat{i}-8\hat{j}-7\hat{k})\ .(2\hat{i}+2\hat{j}+\hat{k})}{3}\right\}\frac{(2\hat{i}+2\hat{j}+\hat{k})}{3} = -\frac{21}{9}(2\hat{i}+2\hat{j}+\hat{k}) = -\frac{7}{3}(2\hat{i}+2\hat{j}+\hat{k})$$

S_4 : Dr's of diagonal BD = a, – a, a or 1,–1,1

Dr's of diagonal AF = – a, a, a or –1, 1, 1

Angle between above diagonals $\cos\theta = \left|\dfrac{-1-1+1}{\sqrt{3}\sqrt{3}}\right| = \dfrac{1}{3}$

विकर्ण BD के दिक् अनुपात = a, – a, a या 1,–1,1

विकर्ण AF के दिक् अनुपात = –a, a, a या –1, 1, 1

उपरोक्त विकर्णो के मध्य कोण $\cos\theta = \left|\dfrac{-1-1+1}{\sqrt{3}\sqrt{3}}\right| = \dfrac{1}{3}$

16. If the distance between points $(\alpha, 5\alpha, 10\alpha)$ from the point of intersection of the line. $\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(2\hat{i} + 4\hat{j} + 12\hat{k})$ and plane $\vec{r}.(\hat{i} - \hat{j} + \hat{k}) = 5$ is 13 units, then value of α may be

यदि रेखा $\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(2\hat{i} + 4\hat{j} + 12\hat{k})$ और समतल $\vec{r}.(\hat{i} - \hat{j} + \hat{k}) = 5$ के प्रतिच्छेद बिन्दु की बिन्दु $(\alpha, 5\alpha, 10\alpha)$ के मध्य दूरी 13 इकाई है तब α का मान हो सकता है

(A) 1 (B*) – 1 (C) 4 (D*) $\dfrac{80}{63}$

Sol. Line : $\dfrac{x-2}{2} = \dfrac{y+1}{4} = \dfrac{z-2}{12} \Rightarrow \dfrac{x-2}{1} = \dfrac{y+1}{2} = \dfrac{z-2}{6} = r$ (Let)

$\Rightarrow$ x = r + 2 $\Rightarrow$ y = 2r – 1 $\Rightarrow$ z = 6r + 2

plane : x – y + z = 5 $\Rightarrow$ r + 2 – (2r – 1) + 6r + 2 = 5 $\Rightarrow$ 5r = 0 $\Rightarrow$ r = 0

$\therefore$ point of intersection is (2, – 1, 2) $\Rightarrow$ $(\alpha - 2)^2 + (5\alpha + 1)^2 + (10\alpha - 2)^2 = 169$

$126\alpha^2 - 34\alpha - 160 = 0 \Rightarrow 63\alpha^2 - 17\alpha - 80 = 0 \Rightarrow \alpha = -1, \dfrac{80}{63}$.

Hindi. रेखा : $\dfrac{x-2}{2} = \dfrac{y+1}{4} = \dfrac{z-2}{12} \Rightarrow \dfrac{x-2}{1} = \dfrac{y+1}{2} = \dfrac{z-2}{6} = r$ (मानाकि)

$\Rightarrow$ x = r + 2 $\Rightarrow$ y = 2r – 1 $\Rightarrow$ z = 6r + 2

समतल : x – y + z = 5 $\Rightarrow$ r + 2 – (2r – 1) + 6r + 2 = 5 $\Rightarrow$ 5r = 0 $\Rightarrow$ r = 0

$\therefore$ प्रतिच्छेद बिन्दु (2, – 1, 2) है। $(\alpha - 2)^2 + (5\alpha + 1)^2 + (10\alpha - 2)^2 = 169$

$\Rightarrow 126\alpha^2 - 34\alpha - 160 = 0 \Rightarrow 63\alpha^2 - 17\alpha - 80 = 0 \Rightarrow \alpha = -1, \dfrac{80}{63}$.

17. A vector $\vec{v} = \lambda(a\hat{j} + b\hat{k})$ is coplanar with the vectors $\hat{i} + \hat{j} - 2\hat{k}$ and $\hat{i} - 2\hat{j} + \hat{k}$ and is orthogonal to the vector $-2\hat{i} + \hat{j} + \hat{k}$. It is given that the length of projection of $\vec{v}$ along the vector $\hat{i} - \hat{j} + \hat{k}$ is equal to $6\sqrt{3}$. Then the value of $\lambda^2 ab$ may be **[16JM120054]**

एक सदिश $\vec{v} = \lambda(a\hat{j} + b\hat{k})$ सदिशों $\hat{i} + \hat{j} - 2\hat{k}$ और $\hat{i} - 2\hat{j} + \hat{k}$ के साथ समतलीय है जो सदिश $-2\hat{i} + \hat{j} + \hat{k}$ के लम्बवत् है। यदि दिया गया है कि $\vec{v}$ के प्रक्षेप की लम्बाई सदिश $\hat{i} - \hat{j} + \hat{k}$ के अनुदिश $6\sqrt{3}$ के बराबर है तब $\lambda^2 ab$ का मान बराबर है

(A) 81 (B) 9 (C) –9 (D*) –81

Sol. $\vec{V} = \lambda(-2\hat{i}+\hat{j}+\hat{k})\times(\hat{i}+\hat{j}-2\hat{k})\times(\hat{i}-2\hat{j}+\hat{k}) = 9\lambda(-\hat{j}+\hat{k}) \Rightarrow \dfrac{(\hat{i}-\hat{j}+\hat{k})\cdot\vec{V}}{|\hat{i}-\hat{j}+\hat{k}|} = \pm 6\sqrt{3} \Rightarrow \lambda = \pm 1$

18. $\hat{a}$ and $\hat{b}$ are two given unit vectors at right angle. The unit vector equally inclined with $\hat{a}$, $\hat{b}$ and $\hat{a}\times\hat{b}$ will be:

$\hat{a}$ एवं $\hat{b}$ दो परस्पर लम्बवत् इकाई सदिश हैं तो $\hat{a}$, $\hat{b}$ और $\hat{a}\times\hat{b}$ के साथ समान कोण बनाने वाला इकाई सदिश होगा–

(A*) $-\dfrac{1}{\sqrt{3}}\left(\hat{a}+\hat{b}+\hat{a}\times\hat{b}\right)$ (B*) $\dfrac{1}{\sqrt{3}}\left(\hat{a}+\hat{b}+\hat{a}\times\hat{b}\right)$

(C) $\dfrac{1}{\sqrt{3}}\left(\hat{a}+\hat{b}-\hat{a}\times\hat{b}\right)$ (D) $-\dfrac{1}{\sqrt{3}}\left(\hat{a}+\hat{b}-\hat{a}\times\hat{b}\right)$

Sol. Let vector is $\vec{\upsilon} = \lambda_1\hat{a}+\lambda_2\hat{b}+\lambda_3(\hat{a}\times\hat{b})$ also $\cos\theta = \dfrac{\vec{\upsilon}.\hat{a}}{|\vec{\upsilon}|\,|\hat{a}|} = \dfrac{\vec{\upsilon}.\hat{b}}{|\vec{\upsilon}|\,|\hat{b}|} = \dfrac{\vec{\upsilon}.(\hat{a}\times\hat{b})}{|\vec{\upsilon}|\,|\hat{a}\times\hat{b}|}$

$\Rightarrow \quad \vec{\upsilon}.\hat{a} = \vec{\upsilon}.\hat{b} = \vec{\upsilon}.(\hat{a}\times\hat{b}) \qquad [\,|\hat{a}\times\hat{b}| = |\hat{a}|\,|\hat{b}|\sin 90^\circ = 1]$

$\Rightarrow \quad \lambda_1 = \lambda_2 = \lambda_3 = \lambda$ (let) $\quad \therefore \quad \vec{\upsilon} = \lambda(\hat{a}+\hat{b}+\hat{a}\times\hat{b})$

$$|\vec{\upsilon}| = \left|\lambda\sqrt{\hat{a}^2+\hat{b}^2+\left(\hat{a}\times\hat{b}\right)^2+2\hat{a}\,.\,\hat{b}+2\hat{b}\,.\,\left(\hat{a}\times\hat{b}\right)+2\left(\hat{a}\times\hat{b}\right).\,\hat{a}}\right| = 1$$

$\Rightarrow \quad \left|\lambda\sqrt{1+1+1}\right| = 1 \quad \Rightarrow \lambda = \pm\dfrac{1}{\sqrt{3}} \quad \therefore \quad \vec{\upsilon} = \pm\dfrac{1}{\sqrt{3}}(\hat{a}+\hat{b}+\hat{a}\times\hat{b})$

Hindi माना अभीष्ठ सदिश $\vec{\upsilon} = \lambda_1\hat{a}+\lambda_2\hat{b}+\lambda_3(\hat{a}\times\hat{b})$ और $\cos\theta = \dfrac{\vec{\upsilon}.\hat{a}}{|\vec{\upsilon}|\,|\hat{a}|} = \dfrac{\vec{\upsilon}.\hat{b}}{|\vec{\upsilon}|\,|\hat{b}|} = \dfrac{\vec{\upsilon}.(\hat{a}\times\hat{b})}{|\vec{\upsilon}|\,|\hat{a}\times\hat{b}|}$

$\Rightarrow \quad \vec{\upsilon}.\hat{a} = \vec{\upsilon}.\hat{b} = \vec{\upsilon}.(\hat{a}\times\hat{b}) \qquad [\,|\hat{a}\times\hat{b}| = |\hat{a}|\,|\hat{b}|\sin 90^\circ = 1]$

$\Rightarrow \quad \lambda_1 = \lambda_2 = \lambda_3 = \lambda$ (let) $\quad \therefore \quad \vec{\upsilon} = \lambda(\hat{a}+\hat{b}+\hat{a}\times\hat{b})$

$$|\vec{\upsilon}| = \left|\lambda\sqrt{\hat{a}^2+\hat{b}^2+\left(\hat{a}\times\hat{b}\right)^2+2\hat{a}\,.\,\hat{b}+2\hat{b}\,.\,\left(\hat{a}\times\hat{b}\right)+2\left(\hat{a}\times\hat{b}\right).\,\hat{a}}\right| = 1$$

$\Rightarrow \quad \left|\lambda\sqrt{1+1+1}\right| = 1 \quad \Rightarrow \lambda = \pm\dfrac{1}{\sqrt{3}} \quad \therefore \quad \vec{\upsilon} = \pm\dfrac{1}{\sqrt{3}}(\hat{a}+\hat{b}+\hat{a}\times\hat{b})$

19. Let $\vec{a} = 2\hat{i}-\hat{j}+\hat{k}$, $\vec{b} = \hat{i}+2\hat{j}-\hat{k}$ and $\vec{c} = \hat{i}+\hat{j}-2\hat{k}$ be three vectors. A vector in the plane of $\vec{b}$ and $\vec{c}$ whose length of projection on $\vec{a}$ is of $\sqrt{\dfrac{2}{3}}$, is **[16JM120055]**

माना $\vec{a} = 2\hat{i}-\hat{j}+\hat{k}$, $\vec{b} = \hat{i}+2\hat{j}-\hat{k}$ तथा $\vec{c} = \hat{i}+\hat{j}-2\hat{k}$ तीन सदिश है। $\vec{b}$ एवं $\vec{c}$ के समतल में एक सदिश जिसका $\vec{a}$ पर प्रक्षेप की लम्बाई, $\sqrt{\dfrac{2}{3}}$ परिमाण का है, होगा–

(A*) 2i + 3j – 3k (B) 2i + 3j + 3k (C*) – 2i – j + 5k (D) i – 5j + 3k

Sol. Let $\vec{r}$ = xi + yj + zk, then $[\vec{r}\ \ \vec{b}\ \ \vec{c}] = 0 \Rightarrow \begin{vmatrix} x & y & z \\ 1 & 2 & -1 \\ 1 & 1 & -2 \end{vmatrix} = 0,\ -3x + y - z = 0$ ....(1)

$\frac{\vec{r}.\vec{a}}{|\vec{a}|} = \pm\frac{\sqrt{2}}{3} \Rightarrow \frac{2x-y+3}{\sqrt{6}} = \pm\sqrt{\frac{2}{3}} \Rightarrow 2x - y + z = \pm 2$ ...(2)

from (1) and (2) $x = \mp 2$; $y - z = \mp 6$ there fore $\vec{r} = \mp 2i + yj + (y \pm 6)k$ (A) & (C) are answer

Hindi माना $\vec{r}$ = xi + yj + zk तब $[\vec{r}\ \ \vec{b}\ \ \vec{c}] = 0 \Rightarrow \begin{vmatrix} x & y & z \\ 1 & 2 & -1 \\ 1 & 1 & -2 \end{vmatrix} = 0,\ -3x + y - z = 0$ ....(1)

$\frac{\vec{r}.\vec{a}}{|\vec{a}|} = \pm\frac{\sqrt{2}}{3} \Rightarrow \frac{2x-y+3}{\sqrt{6}} = \pm\sqrt{\frac{2}{3}} \Rightarrow 2x - y + z = \pm 2$ ...(2)

(1) व (2) से $x = \mp 2$; $y - z = \mp 6$

इसलिये $\vec{r} = \mp 2i + yj + (y \pm 6)k \Rightarrow$ (A) व (C) सही उत्तर है।

20. The position vectors of the angular points of a tetrahedron are $A(3\hat{i}-2\hat{j}+\hat{k}), B(3\hat{i}+\hat{j}+5\hat{k})$, $C(4\hat{i}+3\hat{k})$ and $D(\hat{i})$. Then the acute angle between the lateral face ADC and the base face ABC is :

(A*) $\tan^{-1}\frac{5}{2}$ (B) $\tan^{-1}\frac{2}{5}$ (C) $\cot^{-1}\frac{5}{2}$ (D*) $\cot^{-1}\frac{2}{5}$

किसी चतुष्फलक के शीर्ष बिन्दुओं के स्थिति सदिश $A(3\hat{i}-2\hat{j}+\hat{k}), B(3\hat{i}+\hat{j}+5\hat{k})$, $C(4\hat{i}+3\hat{k})$ तथा $D(\hat{i})$ है, तो ढालु फलक (lateral face) ADC और आधार फलक ABC के बीच न्यून कोण ज्ञात कीजिए ?

(A*) $\tan^{-1}\frac{5}{2}$ (B) $\tan^{-1}\frac{2}{5}$ (C) $\cot^{-1}\frac{5}{2}$ (D*) $\cot^{-1}\frac{2}{5}$

Sol. $\overrightarrow{AD} = -2\hat{i}+2\hat{j}-\hat{k}$, $\overrightarrow{AC} = \hat{i}+2\hat{j}+2\hat{k}$

$\therefore$ vector perpendicular to the face ADC is $= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{vmatrix} = 6\hat{i}+3\hat{j}-6\hat{k} \Rightarrow \overrightarrow{AB} = 3\hat{j}+4\hat{k}$

$\therefore$ A vector perpendicular to the face ABC is $= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 3 & 4 \\ 1 & 2 & 2 \end{vmatrix} = -2\hat{i}+4\hat{j}-3\hat{k}$

$\therefore$ acute angle between the two faces is given by $\cos\theta = \left|\frac{-12+12+18}{\sqrt{36+9+36}\sqrt{4+16+9}}\right| = \frac{2}{\sqrt{29}}$

$\therefore$ $\tan\theta = \frac{5}{2}$ $\therefore$ $\theta = \tan^{-1}\frac{5}{2}$

Hindi $\overrightarrow{AD} = -2\hat{i}+2\hat{j}-\hat{k}$, $\overrightarrow{AC} = \hat{i}+2\hat{j}+2\hat{k}$

$\therefore$ फलक ADC के लम्बवत् सदिश $= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{vmatrix} = 6\hat{i}+3\hat{j}-6\hat{k} \Rightarrow \overrightarrow{AB} = 3\hat{j}+4\hat{k}$

$\therefore$ फलक ABC के लम्बवत् सदिश $= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 3 & 4 \\ 1 & 2 & 2 \end{vmatrix} = -2\hat{i}+4\hat{j}-3\hat{k}$

$\therefore$ दोनों फलकों के बीच न्यूनकोण $\cos\theta = \left|\dfrac{-12+12+18}{\sqrt{36+9+36}\sqrt{4+16+9}}\right| = \dfrac{2}{\sqrt{29}}$

$\therefore$ $\tan\theta = \dfrac{5}{2}$ $\therefore$ $\theta = \tan^{-1}\dfrac{5}{2}$

21. If $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are unit vectors such that $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d}) = \lambda$ and $\vec{a}\,.\,\vec{c} = \dfrac{\sqrt{3}}{2}$, then **[16JM120056]**

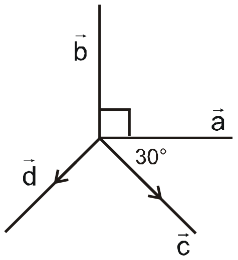
(A*) $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar if $\lambda = 1$
(B) Angle between $\vec{b}$ and $\vec{d}$ is 30° if $\lambda = -1$
(C*) angle between $\vec{b}$ and $\vec{d}$ is 150° if $\lambda = -1$
(D) If $\lambda = 1$ then angle between $\vec{b}$ and $\vec{c}$ is 60°

यदि इकाई सदिश $\vec{a}$, $\vec{b}$, $\vec{c}$ एवं $\vec{d}$ इस प्रकार है कि $(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d}) = \lambda$ तथा $\vec{a}\,.\,\vec{c} = \dfrac{\sqrt{3}}{2}$, तब

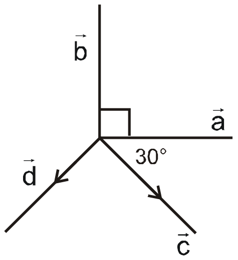
(A*) $\vec{a}$, $\vec{b}$, $\vec{c}$ समतलीय होगें यदि $\lambda = 1$
(B) $\vec{b}$ व $\vec{d}$ के मध्य कोण 30° होगा यदि $\lambda = -1$
(C*) $\vec{b}$ एवं $\vec{d}$ के मध्य कोण 150° होगा यदि $\lambda = -1$
(D) यदि $\lambda = 1$ तब $\vec{b}$ व $\vec{c}$ के मध्य कोण 60° होगा।

Sol. If $\lambda = -1$ then $\vec{a}\perp\vec{b}$, $\vec{c}\perp\vec{d}$ and angle between $\vec{a}\times\vec{b}, \vec{c}\times\vec{d}$ is π

$\angle$ between $\vec{b}$ and $\vec{d}$ = 360° – (90° + 90° + 30°) = 150°

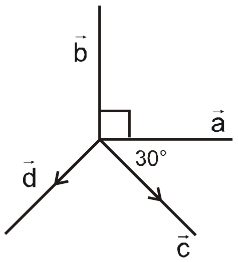


If $(\vec{a}\times\vec{b})\,.\,(\vec{c}\times\vec{d}) = 1$, then following figure is possible then $\angle$ between $\vec{b}$ and $\vec{d} = 30°$

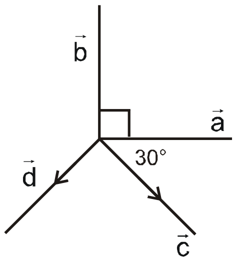


Hindi यदि $\lambda = -1$ तब $\vec{a}\perp\vec{b}, \vec{c}\perp\vec{d}$ तथा $\vec{a}\times\vec{b}, \vec{c}\times\vec{d}$ के मध्य कोण π है।

$\angle$ $\vec{b}$ व $\vec{d}$ के मध्य कोण = 360° – (90° + 90° + 30°) = 150°

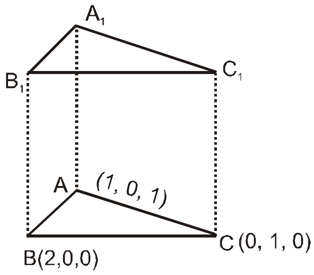


यदि $(\vec{a}\times\vec{b})\,.\,(\vec{c}\times\vec{d}) = 1$,तब निम्न चित्र सम्भव है तब $\angle$ $\vec{b}$ व $\vec{d}$ के मध्य कोण = 30°

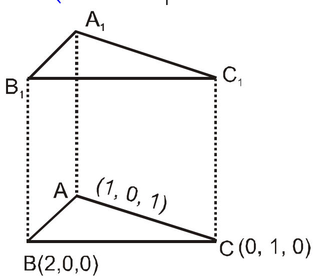


22. The volume of a right triangular prism $ABCA_1B_1C_1$ is equal to 3. If the position vectors of the vertices of the base ABC are A(1, 0, 1), B(2,0, 0) and C(0, 1, 0), then position vectors of the vertex A_1 can be:
एक समकोण त्रिभुजाकार प्रिज्म $ABCA_1B_1C_1$ का आयतन 3 है। यदि आधार ABC के शीर्षो के स्थिति सदिश A(1, 0, 1); B(2,0, 0) और C(0, 1, 0) हैं, तो शीर्ष A_1 का स्थिति सदिश हो सकता है– **[DRN2323]**
(A*) (2, 2, 2) (B) (0, 2, 0) (C) (0, –2, 2) (D*) (0, –2, 0)

Sol. Volume of prism = Area of base ABC × height or $3 = \frac{\sqrt{6}}{2} \times h$
$\Rightarrow \quad h = \sqrt{6}$
Required point A_1 should be just above point A
i.e. line AA_1 is normal to plane ABC and $AA_1 = \sqrt{6}$



Hindi प्रिज्म का आयतन = आधार ABC का क्षैत्रफल × ऊँचाई
or $\quad 3 = \frac{\sqrt{6}}{2} \times h \quad \Rightarrow \quad h = \sqrt{6}$
अभीष्ट बिन्दु A_1, बिन्दु A के ठीक ऊपर होना चाहिए
अर्थात् रेखा AA_1 समतल ABC के लम्बवत् है और $AA_1 = \sqrt{6}$



23. The coplanar points A , B , C , D are (2 – x , 2 , 2) , (2 , 2 – y , 2) , (2 , 2 , 2 – z) and (1 , 1 , 1) respectively, then **[16JM120057]**
(A*) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$ (B*) $\frac{x-1}{x} + \frac{y-1}{y} + \frac{z-1}{z} = 2$
(C) $\frac{1}{1-x} + \frac{1}{1-y} + \frac{1}{1-z} = 1$ (D) $\frac{x}{1-x} + \frac{y}{1-y} + \frac{z}{1-z} + 2 = 0$
यदि चार बिन्दु A , B , C , D क्रमशः (2 – x , 2 , 2) , (2 , 2 – y , 2) , (2 , 2 , 2 – z) और (1 , 1 , 1) समतलीय हैं, तो –

(A*) $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$ (B*) $\frac{x-1}{x}+\frac{y-1}{y}+\frac{z-1}{z}=2$

(C) $\frac{1}{1-x}+\frac{1}{1-y}+\frac{1}{1-z}=1$ (D) $\frac{x}{1-x}+\frac{y}{1-y}+\frac{z}{1-z}+2=0$

Sol. A $(2-x, 2, 2)$, B $(2, 2-y, 2)$, C $(2, 2, 2-z)$, D$(1, 1, 1)$

$\vec{DA}=(1-x)\ \hat{i}+\hat{j}+\hat{k}$; $\vec{DB}=\ \hat{i}+(1-y)\ \hat{j}+\hat{k}$; $\vec{DC}=\ \hat{i}+\ \hat{j}+(1-z)\ \hat{k}$

If four points are coplanar then $[\vec{DA},\ \vec{DB},\ \vec{DC}]=0$

यदि चार बिन्दु समतलीय हैं, तो $[\vec{DA},\ \vec{DB},\ \vec{DC}]=0$

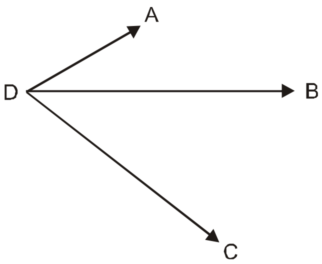
$$\Rightarrow \begin{vmatrix} 1-x & 1 & 1 \\ 1 & 1-y & 1 \\ 1 & 1 & 1-z \end{vmatrix}=0$$

$c_1 \to c_1 - c_2$ and (तथा) $c_2 \to c_2 - c_3$

$$\Rightarrow \begin{vmatrix} 1-x & 1 & 1 \\ 1 & 1-y & 1 \\ 1 & 1 & 1-z \end{vmatrix}=0$$

$-x(-y+yz-z)+1(+yz)=0$ $\Rightarrow xy - xyz + xz + yz = 0$

$xy + yz + zx = xyz$ $\therefore \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1.$



24. Which of the following statement(s) is/are correct :

(A*) If $\vec{a},\vec{b},\vec{c}$ are non–coplanar and $\vec{d}$ is any vector, then

$[\vec{d}\ \ \vec{b}\ \ \vec{c}]\ \vec{a}+[\vec{d}\ \ \vec{c}\ \ \vec{a}]\ \vec{b}+[\vec{d}\ \ \vec{a}\ \ \vec{b}]\ \vec{c}-[\vec{a}\ \ \vec{b}\ \ \vec{c}]\ \vec{d}=\vec{0}$

(B*) If I is incentre of Δ ABC then $|\vec{BC}|\,\vec{IA}+|\vec{CA}|\,\vec{IB}+|\vec{AB}|\,\vec{IC}=\vec{0}$

(C*) Any vector in three dimension can be written as linear combination of three non–coplanar vectors.

(D) In a triangle, if position vector of vertices are $\vec{a},\vec{b},\vec{c}$, then position vector of incentre is $\frac{\vec{a}+\vec{b}+\vec{c}}{3}$.

निम्न में से कौनसा कथन सही है

(A*) यदि $\vec{a},\vec{b},\vec{c}$ असमतलीय है तथा $\vec{d}$ कोई सदिश है तब

$[\vec{d}\ \ \vec{b}\ \ \vec{c}]\ \vec{a}+[\vec{d}\ \ \vec{c}\ \ \vec{a}]\ \vec{b}+[\vec{d}\ \ \vec{a}\ \ \vec{b}]\ \vec{c}-[\vec{a}\ \ \vec{b}\ \ \vec{c}]\ \vec{d}=\vec{0}$

(B*) यदि I, Δ ABC का अन्तःकेन्द्र है तब $|\vec{BC}|\,\vec{IA}+|\vec{CA}|\,\vec{IB}+|\vec{AB}|\,\vec{IC}=\vec{0}$

(C*) त्रिविमीय में कोई सदिश, तीन असमतलीय सदिशों के एक घात संचय के रूप में लिया जा सकता है।

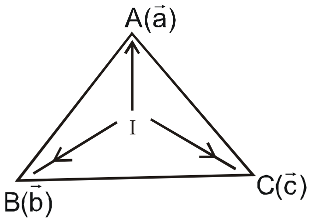
(D) त्रिभुज में यदि शीर्षो के स्थिति सदिश $\vec{a},\vec{b},\vec{c}$ है, तब अन्तकेन्द्र का स्थिति सदिश $\frac{\vec{a}+\vec{b}+\vec{c}}{3}$ है।

Sol. (A) $[\vec{d}\,\vec{b}\,\vec{c}]\,\vec{a}+[\vec{d}\,\vec{c}\,\vec{a}]\,\vec{b}+[\vec{d}\ \ \vec{a}\ \ \vec{b}]\ \vec{c}-[\vec{a}\ \ \vec{b}\ \ \vec{c}]\,\vec{d}$

$=([\vec{b}\ \ \vec{c}\ \ \vec{d}]\ \vec{a}-[\vec{b}\ \ \vec{c}\ \ \vec{a}]\ \vec{d})+([\vec{d}\ \ \vec{a}\ \ \vec{b}]\ \vec{c}-[\vec{d}\ \ \vec{a}\ \ \vec{c}]\ \vec{b})$

$=(\vec{b}\times\vec{c})\times(\vec{a}\times\vec{d})+(\vec{d}\times\vec{a})\times(\vec{c}\times\vec{b})$

$= (\vec{d}\times\vec{a})\times(\vec{b}\times\vec{c}) - (\vec{d}\times\vec{a})\times(\vec{b}\times\vec{c}) = \vec{0}$

(B) 

$I \equiv \dfrac{a\vec{a}+b\vec{b}+c\vec{c}}{a+b+c}$

25. Let $\vec{V} = 2\hat{i}+\hat{j}-\hat{k}$ & $\vec{W} = \hat{i}+3\hat{k}$. If $\vec{U}$ is a unit vector, then the value of the scalar triple product $[\vec{U}\ \vec{V}\ \vec{W}]$ may be : **[16JM120058]**

माना $\vec{V} = 2\hat{i}+\hat{j}-\hat{k}$ तथा $\vec{W} = \hat{i}+3\hat{k}$ है यदि $\vec{U}$ एक इकाई सदिश है तब त्रिक गुणन $[\vec{U}\ \vec{V}\ \vec{W}]$ हो सकता है

(A*) $-\sqrt{59}$ (B*) $\sqrt{10}+\sqrt{6}$ (C*) $\sqrt{59}$ (D) $\sqrt{60}$

Hint : Use formula of scalar triple product अदिश त्रिक गुणन का उपयोग

Sol. $\vec{V} = 2\hat{i}+\hat{j}-\hat{k}$ $\Rightarrow$ $W = \hat{i}+3\hat{k}$ $\Rightarrow$ $[\vec{u}\ \ \vec{v}\ \ \vec{w}] = \vec{u}\,.[(2\hat{i}+\hat{j}-\hat{k})\times(\hat{i}+3\hat{k})]$

$= \vec{u}\,.\,(3\hat{i}-7\hat{j}-\hat{k})$ $|\vec{u}|\,|3\hat{i}-7\hat{j}-\hat{k}|\cos\theta = -\sqrt{59} \le [\vec{u}\vec{v}\vec{w}] \le \sqrt{59}$

26. If $\vec{A}+\vec{B} = \vec{a}$, $\vec{A}.\vec{a} = 1$ and $\vec{A}\times\vec{B} = \vec{b}$, then

यदि $\vec{A}+\vec{B} = \vec{a}$, $\vec{A}.\vec{a} = 1$ तथा $\vec{A}\times\vec{B} = \vec{b}$, तब

(A*) $\vec{A} = \dfrac{\vec{a}\times\vec{b}+\vec{a}}{|\vec{a}|^2}$ (B) $\vec{B} = \dfrac{\vec{a}\times\vec{b}+\vec{a}\,(|\vec{a}|^2-1)}{|\vec{a}|^2}$

(C) $\vec{A} = \dfrac{\vec{b}\times\vec{a}+\vec{a}}{|\vec{a}|^2}$ (D*) $\vec{B} = \dfrac{\vec{b}\times\vec{a}+\vec{a}\,(|\vec{a}|^2-1)}{|\vec{a}|^2}$

Sol. Given दिया गया है $\vec{A}+\vec{B} = \vec{a}$ .....(i)

$\Rightarrow$ $\vec{a}.(\vec{A}+\vec{B}) = \vec{a}.\vec{a}$

$\Rightarrow$ $\vec{a}.\vec{A}+\vec{a}. = \vec{a}.\vec{a}$ $\Rightarrow$ $1+\vec{a}.\vec{B} = |\vec{a}|^2$ $\Rightarrow$ $\vec{a}\,.\,\vec{B} = |\vec{a}|^2-1$ .....(ii)

Given दिया गया है $\vec{A}\times\vec{B} = \vec{b}$ $\Rightarrow$ $\vec{a}\times(\vec{A}\times\vec{B}) = \vec{a}\times\vec{b}$

$\Rightarrow$ $(\vec{a}.\vec{B})\,\vec{A} - (\vec{a}.\vec{A})\,\vec{B} = \vec{a}\times\vec{b} \Rightarrow$ $(|\vec{a}|^2-1)\,\vec{A}-\vec{B} = \vec{a}\times\vec{b}$ .....(iii)

[Using equation (ii)] समीकरण (ii) से

solving equation (i) and (iii) simultaneously, we get $\vec{A} = \dfrac{\vec{a}\times\vec{b}+\vec{a}}{|\vec{a}|^2}$ and $\vec{B} = \dfrac{\vec{b}\times\vec{a}+\vec{a}\,(|\vec{a}|^2-1)}{|\vec{a}|^2}$

27. The line $\dfrac{x-4}{k} = \dfrac{y-2}{1} = \dfrac{z-k^2}{2}$ lies in the plane $2x - 4y + z = 1$. Then the value of k cannot be :

(A) 1 (B*) – 1 (C*) 2 (D*) –2

रेखा $\dfrac{x-4}{k} = \dfrac{y-2}{1} = \dfrac{z-k^2}{2}$ समतल में स्थित $2x - 4y + z = 1$ है, तब k का मान नहीं हो सकता है

(A) 1 (B*) – 1 (C*) 2 (D*) –2

Sol. $(4, 2, k^2)$ lies on $2x - 4y + z = 1$ $\Rightarrow$ $k = \pm1$

Also $(k\hat{i}+\hat{j}+2\hat{k})\,.\,(2\hat{i}-4\hat{j}+\hat{k}) = 0$ $\Rightarrow$ $k = 1$

(4, 2, k^2) समतल पर स्थित है। $2x - 4y + z = 1$ $\Rightarrow$ $k = \pm 1$

अतः $(k\hat{i} + \hat{j} + 2\hat{k}) \cdot (2\hat{i} - 4\hat{j} + \hat{k}) = 0$ $\Rightarrow$ $k = 1$

28. Equation of the plane passing through $A(x_1, y_1, z_1)$ and containing the line $\frac{x-x_2}{d_1} = \frac{y-y_2}{d_2} = \frac{z-z_2}{d_3}$ is

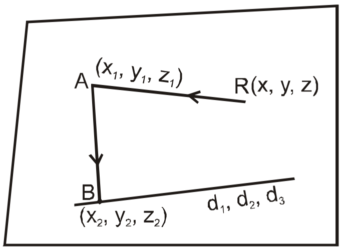
रेखा $\frac{x-x_2}{d_1} = \frac{y-y_2}{d_2} = \frac{z-z_2}{d_3}$ को समाहित करने वाले और बिन्दु $A(x_1, y_1, z_1)$ से गुजरने वाले समतल का समीकरण है –

(A*) $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$ (B*) $\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ x_1-x_2 & y_1-y_2 & z_1-z_2 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$

(C) $\begin{vmatrix} x-d_1 & y-d_2 & z-d_3 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0$ (D) $\begin{vmatrix} x & y & z \\ x_1-x_2 & y_1-y_2 & z_1-z_2 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$

Sol. Vectors $\overrightarrow{AR}$, $\overrightarrow{AB}$ & $\vec{C}$ are coplanar

Equation of the required plane $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$ or $\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ x_1-x_2 & y_1-y_2 & z_1-z_2 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$



$\vec{C} = d_1\hat{i} + d_2\hat{j} + d_3\hat{k}$

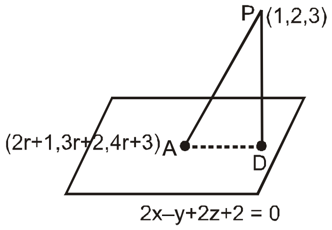
29. A line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ intersects the plane $x - y + 2z + 2 = 0$ at point A. The equation of the straight line passing through A lying in the given plane and at minimum inclination with the given line is/are

एक रेखा $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ समतल $x - y + 2z + 2 = 0$ को बिन्दु A पर प्रतिच्छेद करती है। A से गुजरने वाली सरल रेखा का समीकरण है जो दिए गए समतल में स्थित है और दी गई रेखा के साथ न्यूनतम झुकाव पर है।

(A*) $\frac{x+1}{1} = \frac{y+1}{5} = \frac{z+1}{2}$ (B*) $5x - y + 4 = 0 = 2y - 5z - 3$

(C*) $5x + y - 5z + 1 = 0 = 2y - 5z - 3$ (D*) $\frac{x+2}{1} = \frac{y+6}{5} = \frac{z+3}{2}$

Sol.



$2r + 1 - (3r + 2) + 2(4r + 3) + 2 = 0 \quad \Rightarrow \quad 7r + 7 = 0 \quad \Rightarrow \quad r = -1$

$\therefore$ A(–1, –1, – 1) required line will be projection of given line in the plane foot of $\perp$ of P will be on D

(अभीष्ट रेखा, दी गई रेखा का समतल में प्रक्षेप होगा। P के लम्ब का पाद D पर होगा।)

$\frac{x-1}{1} = \frac{y-2}{-1} = \frac{z-3}{2} = -\left(\frac{1-2+2.3+2}{1^2+(-1)^2+2^2}\right) \quad ; \quad \frac{x-1}{1} = \frac{y-2}{-1} = \frac{z-3}{2} = \frac{-7}{6}$

$x = \frac{-1}{6}\ ;\ y = \frac{19}{6}\ ;\ z = \frac{4}{6} \quad \Rightarrow \quad \frac{x+1}{2} = \frac{y+1}{5} = \frac{z+1}{2} \quad \Rightarrow \quad \frac{x+1}{2} + 1 = \frac{y+1}{5} + 1 = \frac{z+1}{2} + 1$

Also इसलिए, $\frac{x+1}{2} = \frac{y+1}{5} = \frac{z+1}{2} \quad \Rightarrow \quad \frac{x+1}{2} = \frac{y+1}{5}$ and तथा $\frac{y+1}{5} = \frac{z+1}{2}$

$\Rightarrow \quad 5x - y + 4 = 0 = 2y - 5z - 3$

alsoअतः $(5x - y + 4) + (2y - 5z - 3) = 0 = 2y - 5z - 3 \quad \Rightarrow \quad 5x + y - 5z + 1 = 0 = 2y - 5z - 3$

30. If the π-plane $7x + (\alpha + 4)y + 4z - r = 0$ passing through the points of intersection of the planes $2x + 3y - z + 1 = 0$ and $x + y - 2z + 3 = 0$ and is perpendicular to the plane $3x - y - 2z = 4$ and $\left(\frac{12}{\beta}, \frac{-78}{\beta}, \frac{57}{\beta}\right)$ is image of point (1, 1, 1) in π–plane, then **[16JM120059]**

यदि π-समतल $7x + (\alpha + 4)y + 4z - \alpha = 0$ समतलों $2x + 3y - z + 1 = 0$ और $x + y - 2z + 3 = 0$ के प्रतिच्छेद बिन्दुओं से गुजरता है और समतल $3x - y - 2z = 4$ के लम्बवत है तथा बिन्दु (1, 1, 1) का π–समतल में प्रतिबिम्ब $\left(\frac{12}{\beta}, \frac{-78}{\beta}, \frac{57}{\beta}\right)$ है तब –

(A*) $\alpha = 9$ (B) $\beta = -117$ (C) $\alpha = -9$ (D*) $\beta = 117$

Sol. Let the plane is माना समतल है–

$(2x + 3y - z) + 1 + \lambda(x + y - 2z + 3) = 0 \quad ....(1)$

$(2 + \lambda)x + (3 + \lambda)y - (1 + 2\lambda)z + 1 + 3\lambda = 0 \quad \Rightarrow 3(2 + \lambda) - (3 + \lambda) + 2(1 + 2\lambda) = 0$

$+ 6\lambda + 5 = 0 \Rightarrow \lambda = -5/6$

Putting value of λ in (1) (1) में λ का मान रखने पर

$7x + 13y + 4z - 9 = 0 \quad \Rightarrow \quad \alpha = 9$

Now image of (1, 1, 1) in plane π is अब समतल π में (1,1,1) का प्रतिबिम्ब है

$\frac{x-1}{7} = \frac{y-1}{13} = \frac{z-1}{4} = -2\left(\frac{7+13+4-9}{49+169+16}\right) \quad ; \quad \frac{x-1}{7} = \frac{y-1}{13} = \frac{z-1}{4} = -\frac{15}{117}$

$x = \frac{12}{117},\ y = \frac{-78}{117},\ z = \frac{57}{117} \quad \Rightarrow \quad \beta = 117.$

31. The planes $2x - 3y - 7z = 0$, $3x - 14y - 13z = 0$ and $8x - 31y - 33z = 0$ **[16JM120060]**

(A*) pass through origin (B*) intersect in a common line

(C) form a triangular prism (D*) pass through infinite the many points

समतल $2x - 3y - 7z = 0$, $3x - 14y - 13z = 0$ और $8x - 31y - 33z = 0$

(A) मूल बिन्दु से गुजरते हैं। (B) एक उभयनिष्ठ रेखा पर काटते हैं।

(C) एक त्रिकोणीय प्रिज्म बनाते हैं। (D) अनन्त बिन्दुओं से गुजरते हैं।

Sol. $\left.\begin{array}{r} 2x-3y-7z=0 \\ 3x-14y-13z=0 \\ 8x-31y-33z=0 \end{array}\right\}$ obviously all the three planes pass through origin

$$D = \begin{vmatrix} 2 & -3 & -7 \\ 3 & -14 & -13 \\ 8 & -31 & -33 \end{vmatrix} = 2(462-403)+3(-99+104)-7(-93+112) = 118+15-133 = 0$$

From the theory of system of equations $D = D_1 = D_2 = D_3 = 0$

$\Rightarrow$ System of equations has infinite solutions

$\therefore$ hence three planes intersect in a common line

Hindi. $\left.\begin{array}{r} 2x-3y-7z=0 \\ 3x-14y-13z=0 \\ 8x-31y-33z=0 \end{array}\right\}$ स्पष्टतया तीनों समतल मूल बिन्दु से गुजरते हैं।

$$D = \begin{vmatrix} 2 & -3 & -7 \\ 3 & -14 & -13 \\ 8 & -31 & -33 \end{vmatrix} = 2(462-403)+3(-99+104)-7(-93+112) = 118+15-133 = 0$$

समीकरण निकायों के सिद्धान्त से $D = D_1 = D_2 = D_3 = 0$ $\Rightarrow$ समीकरणों के निकाय के अनन्त हल हैं।

$\therefore$ अतः तीनों समतल एक उभयनिष्ठ रेखा पर काटते हैं।

32. If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are the position vectors of the points A, B, C and D respectively in three dimensional space no three of A, B, C, D are collinear and satisfy the relation $3\vec{a}-2\vec{b}+\vec{c}-2\vec{d} = \vec{0}$, then : **[16JM120061]**

(A*) A, B, C and D are coplanar

(B) The line joining the points B and D divides the line joining the point A and C in the ratio 2 : 1.

(C*) The line joining the points A and C divides the line joining the points B and D in the ratio 1 : 1.

(D*) the four vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are linearly dependents.

यदि त्रिविम में $\vec{a}, \vec{b}, \vec{c}$ तथा $\vec{d}$ क्रमशः बिन्दुओं A, B, C एवं D के स्थिति सदिश हैं। बिन्दुओं A, B, C एवं D में से कोई भी तीन संरेखीय नहीं है और सम्बन्ध $3\vec{a}-2\vec{b}+\vec{c}-2\vec{d}=\vec{0}$ को सन्तुष्ट करते हैं तो

(A) A, B, C और D समतलीय है।

(B) B और D को जोड़ने वाली रेखा बिन्दु A तथा C को जोड़ने वाली रेखा को 2 : 1 में विभाजित करती है।

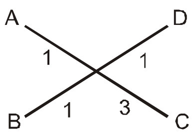
(C) बिन्दु A और C को जोड़ने वाली रेखा B और D को जोड़ने वाली रेखा को 1 : 1 में विभाजित करती है।

(D) चारों सदिश $\vec{a}, \vec{b}, \vec{c}$ और $\vec{d}$ रेखीय परतन्त्र हैं।

Sol. $3\vec{a}-2\vec{b}+\vec{c}-2\vec{d}=0$ sum of coefficient = 0

$\Rightarrow$ $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar

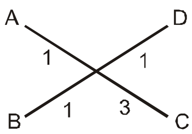
Also $2\vec{b}+2\vec{d} = 3\vec{a}+\vec{c}$ $\Rightarrow$ $\dfrac{\vec{b}+\vec{d}}{2} = \dfrac{3\vec{a}+\vec{c}}{4}$



Hindi $3\vec{a}-2\vec{b}+\vec{c}-2\vec{d}=0$

गुणांकों का योगफल = 0 $\Rightarrow$ $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ समतलीय है।

पुनः $2\vec{b}+2\vec{d} = 3\vec{a}+\vec{c}$ $\Rightarrow$ $\dfrac{\vec{b}+\vec{d}}{2} = \dfrac{3\vec{a}+\vec{c}}{4}$



PART - IV : COMPREHENSION

भाग - IV : अनुच्छेद (COMPREHENSION)

Comprehension # 1
अनुच्छेद # 1

In a parallelogram OABC, vectors $\vec{a}, \vec{b}, \vec{c}$ are respectively the position vectors of vertices A, B, C with reference to O as origin. A point E is taken on the side BC which divides it in the ratio of 2 : 1 internally. Also, the line segment AE intersect the line bisecting the angle O internally in point P. If CP, when extended meets AB in point F. Then

एक समान्तर चतुर्भुज OABC में शीर्षो A, B, C के स्थिति सदिश क्रमशः $\vec{a}, \vec{b}, \vec{c}$ है तथा O मूल बिन्दु है। BC भुजा पर एक बिन्दु E इस प्रकार लिया जाता है कि वह इसे 2 : 1 के अनुपात में अन्तःविभाजित करता है, साथ ही रेखा खण्ड AE कोण O को अन्तःसमद्विभाजित करने वाली रेखा को P बिन्दु पर मिलता है। यदि CP को बढ़ाया जायें तो वह AB को F बिन्दु पर मिलती है, तब

1. The position vector of point P, is
बिन्दु P का स्थिति सदिश है–

(A*) $\dfrac{3|\vec{a}||\vec{c}|}{3|\vec{c}|+2|\vec{a}|}\left\{\dfrac{\vec{a}}{|\vec{a}|}+\dfrac{\vec{c}}{|\vec{c}|}\right\}$

(B) $\dfrac{|\vec{a}||\vec{c}|}{3|\vec{c}|+2|\vec{a}|}\left\{\dfrac{\vec{a}}{|\vec{a}|}+\dfrac{\vec{c}}{|\vec{c}|}\right\}$

(C) $\dfrac{2|\vec{a}||\vec{c}|}{3|\vec{c}|+2|\vec{a}|}\left\{\dfrac{\vec{a}}{|\vec{a}|}+\dfrac{\vec{c}}{|\vec{c}|}\right\}$

(D) $\dfrac{3|\vec{a}||\vec{c}|}{3|\vec{c}|+2|\vec{a}|}\left\{\dfrac{\vec{a}}{|\vec{a}|}-\dfrac{\vec{c}}{|\vec{c}|}\right\}$

2. The position vector of point F, is
बिन्दु F का स्थिति सदिश है–

(A*) $\vec{a}+\dfrac{1}{3}\dfrac{|\vec{a}|}{|\vec{c}|}\vec{c}$ (B) $\vec{a}+\dfrac{|\vec{a}|}{|\vec{c}|}\vec{c}$ (C) $\vec{a}+\dfrac{2|\vec{a}|}{|\vec{c}|}\vec{c}$ (D) $\vec{a}-\dfrac{|\vec{a}|}{|\vec{c}|}\vec{c}$

3. The vector $\overrightarrow{AF}$, is given by
सदिश $\overrightarrow{AF}$ निम्न द्वारा दिया जाता है–

(A) $-\dfrac{|\vec{a}|}{|\vec{c}|}\vec{c}$ (B) $\dfrac{|\vec{a}|}{|\vec{c}|}\vec{c}$ (C) $\dfrac{2|\vec{a}|}{|\vec{c}|}\vec{c}$ (D*) $\dfrac{1}{3}\dfrac{|\vec{a}|}{|\vec{c}|}\vec{c}$

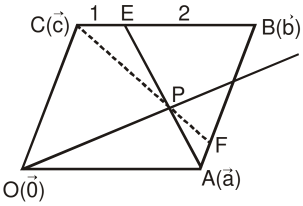
Sol. for 1 to 3

$$E = \frac{2\vec{c}+\vec{b}}{3}$$

equation of OP $\vec{r} = \lambda\left(\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{c}}{|\vec{c}|}\right)$ ...(1)

Let P divide EA in μ : 1 $\Rightarrow$ $P\left[\frac{\mu\vec{a} + \frac{2\vec{c}+\vec{b}}{3}}{\mu+1}\right]$

P lies on (1)



$\frac{\mu\vec{a} + \frac{2\vec{c}+\vec{b}}{3}}{\mu+1} = \lambda\left(\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{c}}{|\vec{c}|}\right)$ $\Rightarrow$ $\vec{a}+\vec{c}=\vec{b}$ $\Rightarrow$ $\frac{\mu\vec{a} + \frac{3\vec{c}+\vec{a}}{3}}{\mu+1} = \lambda\left(\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{c}}{|\vec{c}|}\right)$

Comparing coefficient of $\vec{a}$ and $\vec{c}$

$\frac{\mu + \frac{1}{3}}{\mu = 1} = \frac{\lambda}{|\vec{a}|}$ ...(2) ; and $\frac{1}{\mu+1} = \frac{\lambda}{|\vec{c}|}$ ...(3)

divided (2) by (3)

$\mu + \frac{1}{3} = \frac{|\vec{c}|}{|\vec{a}|}$ $\Rightarrow$ $\mu = \frac{|\vec{c}|}{|\vec{a}|} - \frac{1}{3}$

Put in (3) $\Rightarrow$ $\frac{1}{\frac{|c|}{|a|} + \frac{2}{3}} = \frac{\lambda}{|\vec{c}|}$ $\Rightarrow$ $\lambda = \frac{3|\vec{a}|\,|\vec{c}|}{c|\vec{c}| + 2|\vec{a}|}$

So position vector of P $\vec{r} = \frac{3|\vec{a}|\,|\vec{c}|}{3|\vec{c}| + 2|\vec{a}|}\left(\frac{a}{|\vec{a}|} + \frac{c}{|\vec{c}|}\right)$. Now for solution of 4 equation of AB

$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}) = \vec{a} + \lambda(\vec{c})$ ...(4)

equation of CP $\Rightarrow$ $\vec{r} = \vec{c} + \mu\left(\frac{3|c|\vec{a}}{3|\vec{c}| + 2|\vec{a}|} + \frac{3|a|\vec{c}}{3|c| + 2|\vec{a}|} - \vec{c}\right)$

$\vec{r} = \vec{c} + \mu\left[\frac{3|c|\vec{a} + 3|a|c - 3|c|c - 2|a|\vec{c}}{3|c| + 2|a|}\right] \Rightarrow r = \vec{c} + \mu\left[\frac{3|c|\vec{a} + |a|\vec{c} - 3|c|\vec{c}}{3|c| + 2|a|}\right]$ ..(5)

Comparing (4) and (5)

$\lambda = 1 + \frac{\mu|a| - 3\mu|c|}{3|c| + 2|a|}$ $\Rightarrow$ $\lambda = \frac{3|\vec{c}| + 2|\vec{a}| + \mu|a| - 3\mu|\vec{c}|}{3|\vec{c}| + 2|\vec{a}|}$ ...(6)

$\mu = \frac{3|\vec{c}| + 2|\vec{a}|}{3|\vec{c}|}$

Put value of μ in equation 6

$\lambda = 1 + \frac{\mu(|\vec{a}| - 3|\vec{c}|)}{3|\vec{c}| + 2|\vec{a}|}$ $\Rightarrow$ $\lambda = 1 + \frac{|a| - 3|\vec{c}|}{3|c|} = \frac{1}{3}\frac{|a|}{|c|}$

So position vector of F is $= \vec{a} + \frac{1}{3}\frac{|a|}{|c|}\vec{c}$ $\Rightarrow$ Solution – 5

$\vec{A}\ \ \vec{F}$ = p.v. of F – p.v. of A $= \vec{a} + \frac{1}{3}\frac{|a|}{|c|}\vec{c} - \vec{a} = \frac{1}{3}\frac{|a|}{|c|}\vec{c}$

Sol. for 1 to 3

$E = \frac{2\vec{c}+\vec{b}}{3}$ $\Rightarrow$ OP का समीकरण $\vec{r} = \lambda\left(\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{c}}{|\vec{c}|}\right)$ ...(1)

माना P, EA को $\mu : 1$ में विभाजित करता है।

$P\left[\frac{\mu\vec{a} + \frac{2\vec{c}+\vec{b}}{3}}{\mu+1}\right] \Rightarrow$ P, (1) पर स्थित है $\Rightarrow \frac{\mu\vec{a} + \frac{2\vec{c}+\vec{b}}{3}}{\mu+1} = \lambda\left(\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{c}}{|\vec{c}|}\right)$

$\vec{a}+\vec{c}=\vec{b} \Rightarrow \frac{\mu\vec{a} + \frac{3\vec{c}+\vec{a}}{3}}{\mu+1} = \lambda\left(\frac{\vec{a}}{|\vec{a}|} + \frac{\vec{c}}{|\vec{c}|}\right) \Rightarrow$ $\vec{a}$ तथा $\vec{c}$ के गुणांक की तुलना करने पर

$\frac{\mu + \frac{1}{3}}{\mu = 1} = \frac{\lambda}{|\vec{a}|}$ ...(2) तथा $\frac{1}{\mu+1} = \frac{\lambda}{|\vec{c}|}$ ...(3)

(2) को (3) से विभाजित करने पर

$\mu + \frac{1}{3} = \frac{|\vec{c}|}{|\vec{a}|}$ $\Rightarrow$ $\mu = \frac{|\vec{c}|}{|\vec{a}|} - \frac{1}{3}$

(3) में रखने पर $\Rightarrow$ $\frac{1}{\frac{|c|}{|a|} + \frac{2}{3}} = \frac{\lambda}{|\vec{c}|}$ $\Rightarrow$ $\lambda = \frac{3|\vec{a}|\,|\vec{c}|}{c|\vec{c}|+2|\vec{a}|}$

इसलिए P का स्थिति सदिश $\Rightarrow$ $\vec{r} = \frac{3|\vec{a}|\,|\vec{c}|}{3|\vec{c}|+2|\vec{a}|}\left(\frac{a}{|\vec{a}|} + \frac{c}{|\vec{c}|}\right)$

अब 4 के हल के लिए AB का समीकरण $\vec{r} = \vec{a} + \lambda(\vec{b}-\vec{a}) = \vec{a} + \lambda(\vec{c})$ ...(4)

CP का समीकरण

$\vec{r} = \vec{c} + \mu\left(\frac{3|c|\vec{a}}{3|\vec{c}|+2|\vec{a}|} + \frac{3|a|\vec{c}}{3|c|+2|\vec{a}|} - \vec{c}\right) \Rightarrow \vec{r} = \vec{c} + \mu\left[\frac{3|c|\vec{a}+3|a|c-3|c|c-2|a|\vec{c}}{3|c|+2|a|}\right]$

$r = \vec{c} + \mu\left[\frac{3|c|\vec{a}+|a|\vec{c}-3|c|\vec{c}}{3|c|+2|a|}\right]$ ..(5)

(4) और (5) की तुलना करने पर

$\lambda = 1 + \frac{\mu|a|-3\mu|c|}{3|c|+2|a|}$ $\Rightarrow$ $\lambda = \frac{3|\vec{c}|+2|\vec{a}|+\mu|a|-3\mu|\vec{c}|}{3|\vec{c}|+2|\vec{a}|}$ ...(6)

$\mu = \frac{3|\vec{c}|+2|\vec{a}|}{3|\vec{c}|}$

समीकरण 6 में μ का मान रखने पर

$\lambda = 1 + \frac{\mu(|\vec{a}|-3|\vec{c}|)}{3|\vec{c}|+2|\vec{a}|} \Rightarrow \lambda = 1 + \frac{|a|-3|\vec{c}|}{3|c|} = \frac{1}{3}\frac{|a|}{|c|}$

इसलिए F का स्थिति सदिश $= \vec{a} + \frac{1}{3}\frac{|a|}{|c|}\vec{c}$ $\Rightarrow$ हल – 5

$$\vec{AF} = \text{p.v. of F} - \text{p.v. of A} = \vec{a} + \frac{1}{3}\frac{|a|}{|c|}\vec{c} - \vec{a} = \frac{1}{3}\frac{|a|}{|c|}\vec{c}$$

Comprehension # 2

Let $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ be two planes, where $d_1, d_2 > 0$. Then origin lies in acute angle if $a_1a_2 + b_1b_2 + c_1c_2 < 0$ and origin lies in obtuse angle if $a_1a_2 + b_1b_2 + c_1c_2 > 0$.
Further point (x_1, y_1, z_1) and origin both lie either in acute angle or in obtuse angle ,
if $(a_1x_1 + b_1y_1 + c_1z_1 + d_1)(a_2x_1 + b_2y_1 + c_2z_1 + d_2) > 0$, one of (x_1, y_1, z_1) and origin lie in acute angle and the other in obtuse angle, if $(a_1x_1 + b_1y_1 + c_1z_1 + d_1)(a_2x_1 + b_2y_1 + c_2z_1 + d_2) < 0$

4. Given the planes $2x + 3y - 4z + 7 = 0$ and $x - 2y + 3z - 5 = 0$, if a point P is $(1, -2, 3)$ and O is origin, then
(A) O and P both lie in acute angle between the planes
(B*) O and P both lie in obtuse angle between the planes
(C) O lies in acute angle, P lies in obtuse angle.
(D) O lies in obtue angle, P lies in acute angle.

Sol. **(B)**
Equation of the second plane is $-x + 2y - 3z + 5 = 0 \Rightarrow 2(-1) + 3 . 2 + (-4)(-3) > 0$
$\therefore$ O lies in obtuse angle $(2 \times 1 + 3(-2) - 4 \times 3 + 7)(-1 + 2(-2) - 3 \times 3 + 5)$
$= (2 - 6 - 12 + 7)(-1 - 4 - 9 + 5) > 0$ $\therefore$ P lies in obtuse angle.

5. Given the planes $x + 2y - 3z + 5 = 0$ and $2x + y + 3z + 1 = 0$. If a point P is $(2, -1, 2)$ and O is origin, then
(A) O and P both lie in acute angle between the planes
(B) O and P both lie in obtuse angle between the planes
(C*) O lies in acute angle, P lies in obtuse angle.
(D) O lies in obtue angle, P lies in acute angle.

Sol. **(C)**
$1 \times 2 + 2 \times 1 - 3 \times 3 < 0 \therefore$ O lies in acute angle.
Also $(2 + 2(-1) - 3(2) + 5)(2 \times 2 - 1 + 3 \times 2 + 1) = (-1)(10) < 0$ $\therefore$ P lies in obtuse angle.

6. Given the planes $x + 2y - 3z + 2 = 0$ and $x - 2y + 3z + 7 = 0$, if the point P is $(1, 2, 2)$ and O is origin, then
(A*) O and P both lie in acute angle between the planes
(B) O and P both lie in obtuse angle between the planes
(C) O lies in acute angle, P lies in obtuse angle.
(D) O lies in obtue angle, P lies in acute angle.

Sol. **(A)**
$1 - 4 - 9 < 0$ $\therefore$ O lies in acute angle.
Further $(1 + 4 - 6 + 2)(1 - 4 + 6 + 7) > 0$ $\therefore$ The point P lies in acute angle.

अनुच्छेद # 2

माना कि $a_1x + b_1y + c_1z + d_1 = 0$ तथा $a_2x + b_2y + c_2z + d_2 = 0$ दो समतल हैं जहाँ $d_1, d_2 > 0$ है। तब मूल बिन्दु न्यूनकोण में स्थित होगा यदि $a_1a_2 + b_1b_2 + c_1c_2 < 0$ तथा मूल बिन्दु अधिक कोण में स्थित होगा यदि $a_1a_2 + b_1b_2 + c_1c_2 > 0$ है।
इसके अलावा बिन्दु (x_1, y_1, z_1) तथा मूल बिन्दु दोनों, या तो न्यूनकोण में या अधिक कोण में होगें यदि
$(a_1x_1 + b_1y_1 + c_1z_1 + d_1)(a_2x_1 + b_2y_1 + c_2z_1 + d_2) > 0$,
पुनः बिन्दु (x_1, y_1, z_1) तथा मूल बिन्दु में से कोई एक, न्यूनकोण में तथा दूसरा अधिक कोण में होगा, यदि $(a_1x_1 + b_1y_1 + c_1z_1 + d_1)(a_2x_1 + b_2y_1 + c_2z_1 + d_2) < 0$.

4. समतल $2x + 3y - 4z + 7 = 0$ तथा $x - 2y + 3z - 5 = 0$ दिये गये हैं यदि बिन्दु $P(1, -2, 3)$ है और O मूल बिन्दु है, तब–

(A) O तथा P दोनों समतलों के मध्य न्यूनकोण में स्थित हैं
(B*) O तथा P दोनों समतलों के मध्य अधिक कोण में स्थित हैं
(C) O न्यूनकोण में तथा P अधिक कोण में स्थित हैं
(D) O अधिक कोण में तथा P न्यूनकोण में स्थित हैं

Sol. **(B)**
दूसरे समतल का समीकरण $-x + 2y - 3z + 5 = 0$ है $2(-1) + 3 . 2 + (-4)(-3) > 0$
$\therefore$ O अधिक कोण में स्थित है। $(2 \times 1 + 3(-2) - 4 \times 3 + 7)(-1 + 2(-2) - 3 \times 3 + 5)$
$= (2 - 6 - 12 + 7)(-1 - 4 - 9 + 5) > 0$ $\therefore$ P अधिक कोण में स्थित है।

5. समतल $x + 2y - 3z + 5 = 0$ तथा $2x + y + 3z + 1 = 0$ दिये गये हैं। यदि एक बिन्दु P(2, –1, 2) है और O मूल बिन्दु है, तब–
(A) O तथा P दोनों समतलों के मध्य न्यूनकोण में स्थित हैं।
(B) O तथा P दोनों समतलों के मध्य अधिक कोण में स्थित हैं।
(C*) O न्यूनकोण में तथा P अधिक कोण में स्थित हैं।
(D) O अधिक कोण में तथा P न्यूनकोण में स्थित हैं।

Sol. **(C)**
$1 \times 2 + 2 \times 1 - 3 \times 3 < 0 \therefore$ O न्यूनकोण में स्थित है।
एवं $(2 + 2(-1) - 3(2) + 5)(2 \times 2 - 1 + 3 \times 2 + 1) = (-1)(10) < 0$ $\therefore$ P अधिक कोण में स्थित है।

6. समतल $x + 2y - 3z + 2 = 0$ तथा $x - 2y + 3z + 7 = 0$ दिये गऐ हैं यदि बिन्दु P(1, 2, 2) है और O मूल बिन्दु है, तब–
(A*) O तथा P दोनों समतलों के मध्य न्यूनकोण में स्थित हैं
(B) O तथा P दोनों समतलों के मध्य अधिक कोण में स्थित हैं
(C) O न्यूनकोण में तथा P अधिक कोण में स्थित हैं
(D) O अधिक कोण में तथा P न्यूनकोण में स्थित हैं

Sol. **(A)**
$1 - 4 - 9 < 0$ $\therefore$ O न्यूनकोण में स्थित है।
पुनः $(1 + 4 - 6 + 2)(1 - 4 + 6 + 7) > 0$ $\therefore$ बिन्दु P न्यूनकोण में स्थित है।

Comprehension # 3

If $\vec{a}, \vec{b}, \vec{c}$ & $\vec{a}', \vec{b}', \vec{c}'$ are two sets of non-coplanar vectors such that $\vec{a}.\vec{a}' = \vec{b}.\vec{b}' = \vec{c}.\vec{c}' = 1$, then the two systems are called Reciprocal System of vectors and $\vec{a}' = \frac{\vec{b} \times \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$, $\vec{b}' = \frac{\vec{c} \times \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$ and $\vec{c}' = \frac{\vec{a} \times \vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$.

यदि $\vec{a}, \vec{b}, \vec{c}$ व $\vec{a}', \vec{b}', \vec{c}'$ दो असमतलीय सदिश के दो समुच्चय इस प्रकार है कि $\vec{a}.\vec{a}' = \vec{b}.\vec{b}' = \vec{c}.\vec{c}' = 1$, तब दोनों निकायो को सदिशों का व्युत्क्रम निकाय कहते है। तथा $\vec{a}' = \frac{\vec{b} \times \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$, $\vec{b}' = \frac{\vec{c} \times \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$ और $\vec{c}' = \frac{\vec{a} \times \vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$.

7. Find the value of $\vec{a} \times \vec{a}' + \vec{b} \times \vec{b}' + \vec{c} \times \vec{c}'$.
$\vec{a} \times \vec{a}' + \vec{b} \times \vec{b}' + \vec{c} \times \vec{c}'$ का मान होगा
(A*) $\vec{0}$ (B) $\vec{a} + \vec{b} + \vec{c}$ (C) $\vec{a} - \vec{b} + \vec{c}$ (D) $\vec{a} + \vec{b} - \vec{c}$

Sol. We have हम जानते है : $\vec{a}' = \lambda(\vec{b} \times \vec{c})$, $\vec{b}' = \lambda(\vec{c} \times \vec{a})$ and तथा $\vec{c}' = \lambda(\vec{a} \times \vec{b})$, where जहाँ $\lambda = \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}$

$\vec{a} \times \vec{a}' = \vec{a} \times \lambda(\vec{b} \times \vec{c}) = \lambda\{\vec{a} \times (\vec{b} \times \vec{c})\} = \lambda\{(\vec{a} . \vec{c})\ \vec{b} - (\vec{a} . \vec{b})\ \vec{c}\}$
$\vec{b} \times \vec{b}' = \vec{b} \times \lambda(\vec{c} \times \vec{a}) = \lambda\{\vec{b} \times (\vec{c} \times \vec{a})\} = \lambda\{(\vec{b} . \vec{a})\ \vec{c} - (\vec{b} . \vec{c})\ \vec{a}\}$
तथा and $\vec{c} \times \vec{c}' = \vec{c} \times \lambda(\vec{a} \times \vec{b}) = \lambda\{\vec{c} \times (\vec{a} \times \vec{b})\} = \lambda\{(\vec{c} . \vec{b})\ \vec{a} - (\vec{c} . \vec{a})\ \vec{b}\}$
$\therefore \vec{a} \times \vec{a}' + \vec{b} \times \vec{b}' + \vec{c} \times \vec{c}' = \lambda\{(\vec{a} . \vec{c})\ \vec{b} - (\vec{a} . \vec{b})\ \vec{c}\} + \lambda\{(\vec{b} . \vec{a})\ \vec{c} - (\vec{b} . \vec{c})\ \vec{a}\} + \lambda\{(\vec{c} . \vec{b})\ \vec{a} - (\vec{c} . \vec{a})\ \vec{b}\}$
$= \lambda[(\vec{a} . \vec{c})\ \vec{b} - (\vec{a} . \vec{b})\ \vec{c} + (\vec{b} . \vec{a})\ \vec{c} - (\vec{b} . \vec{c})\ \vec{a} + (\vec{c} . \vec{b})\ \vec{a} - (\vec{c} . \vec{a})\ \vec{b}]$

$= \lambda\ [(\vec{a}\ .\ \vec{c})\ \vec{b} - (\vec{a}\ .\ \vec{b})\ \vec{c} + (\vec{a}\ .\ \vec{b})\ \vec{c} - (\vec{b}\ .\ \vec{c})\ \vec{a} + (\vec{b}\ .\ \vec{c})\ \vec{a} - (\vec{a}\ .\ \vec{c})\ \vec{b}] = \lambda\vec{0} = \vec{0}$

8. Find value of λ such that $\vec{a}' \times \vec{b}' + \vec{b}' \times \vec{c}' + \vec{c}' \times \vec{a}' = \lambda \dfrac{\vec{a}+\vec{b}+\vec{c}}{[\vec{a}\vec{b}\vec{c}]}$.

λ का मान ज्ञात कीजिए जबकि $\vec{a}' \times \vec{b}' + \vec{b}' \times \vec{c}' + \vec{c}' \times \vec{a}' = \lambda \dfrac{\vec{a}+\vec{b}+\vec{c}}{[\vec{a}\vec{b}\vec{c}]}$.

(A) – 1 (B*) 1 (C) 2 (D) – 2

Sol. $\vec{a}' \times \vec{b}' = \dfrac{(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})}{[\vec{a}\vec{b}\vec{c}]^2} = \dfrac{\vec{c}}{[\vec{a}\vec{b}\vec{c}]}$ $\therefore$ $\vec{a}' \times \vec{b}' + \vec{b}' \times \vec{c}' + \vec{c}' \times \vec{a}' = \dfrac{\vec{a}+\vec{b}+\vec{c}}{[\vec{a}\vec{b}\vec{c}]}$ so इसलिए $\lambda = 1$

9. If $[(a' \times b') \times (b' \times c')\ \ (b' \times c') \times (c' \times a')\ \ (c' \times a') \times (a' \times b')] = [abc]^n$, then find n.

यदि $[(a' \times b') \times (b' \times c')\ (b' \times c') \times (c' \times a')\ \ (c' \times a') \times (a' \times b')] = [abc]^n$, हो तो n का मान ज्ञात कीजिए।

(A*) n = – 4 (B) n = 4 (C) n = – 3 (D) n = 3

Sol. $(a' \times b') \times (b' \times c') = \dfrac{\vec{c}\times\vec{a}}{[\vec{a}\vec{b}\vec{c}]^2} \Rightarrow \left[\dfrac{\vec{c}\times\vec{a}}{[\vec{a}\vec{b}\vec{c}]^2}\ \dfrac{\vec{a}\times\vec{b}}{[\vec{a}\vec{b}\vec{c}]^2}\ \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\vec{b}\vec{c}]^2}\right] = \dfrac{[\vec{a}\vec{b}\vec{c}]^2}{[\vec{a}\vec{b}\vec{c}]^6} = [\vec{a}\vec{b}\vec{c}]^{-4}$ $\therefore$ n = – 4

Comprehension # 4

The vertices of square pyramid are A(0, 0, 0), B(4, 0, 0), C(4, 0, 4), D(0, 0, 4) and E(2, 6, 6)

10. Volume of the pyramid is : **[16JM120062]**

(A*) 32 (B) 16 (C) 8 (D) 4

11. Centroids of triangular faces of square pyramid are

(A) Non-coplanar (B) Coplanar but the plane is not parallel to base plane

(C*) Coplanar & plane is parallel to base plane (D) Co-linear

12. The distance of the plane EBC from ortho-centre of ΔABD is : **[16JM120064]**

(A) 2 (B) 5 (C*) $\dfrac{12}{\sqrt{10}}$ (D) $\sqrt{10}$

Sol. **(10)** Volume of pyramid $= \dfrac{1}{3} \times$ base area $\times$ height $= \dfrac{1}{3} \times 16 \times 6 = 32$

(11) Co-ordinates of centroid of faces EAB, EBC, ECD & EDA are (2, 2, 2) $\left(\dfrac{10}{3}, 2, \dfrac{10}{3}\right)$, $\left(2, 2, \dfrac{14}{3}\right)$, $\left(\dfrac{2}{3}, 2, \dfrac{10}{3}\right)$. y-co-ordinate of each point is 2. hence these points are co-planar & plane is parallel to base plane.

(12) ortho-centre of ΔABD is (0, 0, 0) equation of plane EBC is $\begin{vmatrix} x-4 & y & z \\ -2 & 6 & 6 \\ 0 & 0 & 4 \end{vmatrix} = 0$

$4.((x - 4).6 + 2y) = 0 \Rightarrow 6x - 24 + 2y = 0 \Rightarrow 3x + y - 12 = 0$

$d = \left|\dfrac{0+0+21}{\sqrt{3^2+1}}\right| = \dfrac{12}{\sqrt{10}}$

अनुच्छेद # 4

वर्ग पिरामिड के शीर्ष A(0, 0, 0), B(4, 0, 0), C(4, 0, 4), D(0, 0, 4) और E(2, 6, 6) है।

10. पिरामिड का आयतन
(A*) 32 (B) 16 (C) 8 (D) 4

11. वर्ग पिरामिड के त्रिभुजाकार फलकों का केन्द्रक है –
(A) असमतलीय
(B) समतलीय परन्तु समतल, आधार समतल के समान्तर नहीं है।
(C) समतलीय और समतल, आधार समतल के समान्तर है।
(D) संरेखीय

12. ΔABD के लम्बकेन्द्र से समतल EBC की दूरी है –
(A) 2 (B) 5 (C*) $\frac{12}{\sqrt{10}}$ (D) $\sqrt{10}$

Sol. **(10)** पिरामिड का आयतन $= \frac{1}{3}$ × आधार का क्षेत्रफल × ऊँचाई $= \frac{1}{3} \times 16 \times 6 = 32$

(11) फलकों EAB, EBC, ECD तथा EDA के केन्द्रक क्रमशः $(2, 2, 2)$ $\left(\frac{10}{3}, 2, \frac{10}{3}\right)$, $\left(2, 2, \frac{14}{3}\right)$, $\left(\frac{2}{3}, 2, \frac{10}{3}\right)$ होगें। प्रत्येक बिन्दु का y-निर्देशांक 2 है अतः ये बिन्दु समतलीय है तथा समतल आधार समतल के समान्तर है।

(12) ΔABD का लम्बकेन्द्र $(0, 0, 0)$ है। समतल EBC का समीकरण $\begin{vmatrix} x-4 & y & z \\ -2 & 6 & 6 \\ 0 & 0 & 4 \end{vmatrix} = 0$ है।

$4.((x - 4).6 + 2y) = 0 \Rightarrow 6x - 24 + 2y = 0 \Rightarrow 3x + y - 12 = 0$

$d = \left| \frac{0+0+21}{\sqrt{3^2+1}} \right| = \frac{12}{\sqrt{10}}$

Comprehension # 5

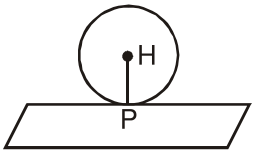
General equation of a sphere is given by $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, where $(-u, -v, -w)$ is the centre and $\sqrt{u^2+v^2+w^2-d}$ is the radius of the sphere.
Let P be a any plane and F is the foot of perpendicular from centre(C) of the sphere to this plane.
If $CF > \sqrt{u^2+v^2+w^2-d}$ then plane P neither touches nor cuts the sphere.
If $CF = \sqrt{u^2+v^2+w^2-d}$ then plane P touches the sphere.
If $CF < \sqrt{u^2+v^2+w^2-d}$ then intersection of plane P and sphere is a circle with radius $= \sqrt{u^2+v^2+w^2-d-(CF)^2}$

अनुच्छेद # 5

गोले का समीकरण जो $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ से दिया जाता है, जहाँ केन्द्र $(-u, -v, -w)$ और गोले कि त्रिज्या $\sqrt{u^2+v^2+w^2-d}$ होती है।
माना P कोई समतल है तथा F, गोले के केन्द्र से समतल पर लम्बपाद के निर्देशांक है।
यदि $CF > \sqrt{u^2+v^2+w^2-d}$ तब समतल P, गोले को न तो स्पर्श करता है और न ही प्रतिच्छेद करता है।
यदि $CF = \sqrt{u^2+v^2+w^2-d}$ तब समतल P, गोले को स्पर्श करता है।
यदि $CF < \sqrt{u^2+v^2+w^2-d}$ तब समतल P, तथा गोले के प्रतिच्छेदन से एक वृत्त है जिसकी ध्त्रिज्या $= \sqrt{u^2+v^2+w^2-d-(CF)^2}$

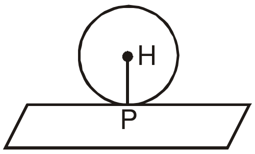
13. Find the equation of the sphere having centre at (1, 2, 3) and touching the plane $x + 2y + 3z = 0$.
समतल $x + 2y + 3z = 0$ को स्पर्श करने वाले उस गोले का समीकरण ज्ञात कीजिए जिसका केन्द्र (1, 2, 3) है।
(A*) $x^2 + y^2 + z^2 - 2x - 4y - 6z = 0$
(B) $x^2 + y^2 + z^2 - 2x + 4y - 6z = 0$
(C) $x^2 + y^2 + z^2 - 2x - 4y + 6z = 0$
(D) $x^2 + y^2 + z^2 + 2x - 4y - 6z = 0$

Solution : Given plane is $x + 2y + 3z = 0$ ..... (1)
Let H be the centre of the required sphere.



Given $H \equiv (1, 2, 3)$
Radius of the sphere,
HP = length of perpendicular from H to plane (1)
$$= \frac{|1+2\times2+3\times3|}{\sqrt{14}} = \sqrt{14}$$
Equation of the required sphere is $(x-1)^2 + (y-2)^2 + (z-3)^2 = 14$
or $x^2 + y^2 + z^2 - 2x - 4y - 6z = 0$

हल दिया गया समतल $x + 2y + 3z = 0$ है ..... (1)
माना अभीष्ट गोले का केन्द्र H है।



दिया है, $H \equiv (1, 2, 3)$
गोले की त्रिज्या HP,
$$= H \text{ की समतल (1) से लम्बवत दूरी} = \frac{|1+2\times2+3\times3|}{\sqrt{14}} = \sqrt{14}$$
अतः अभीष्ट गोले का समीकरण होगा।
$(x-1)^2 + (y-2)^2 + (z-3)^2 = 14$ या $x^2 + y^2 + z^2 - 2x - 4y - 6z = 0$

14. A variable plane passes through a fixed point (1, 2, 3). The locus of the foot of the perpendicular drawn from origin to this plane is:
(A*) $x^2 + y^2 + z^2 - x - 2y - 3z = 0$
(B) $x^2 + 2y^2 + 3z^2 - x - 2y - 3z = 0$
(C) $x^2 + 4y^2 + 9z^2 + x + 2y + 3 = 0$
(D) $x^2 + y^2 + z^2 + x + 2y + 3z = 0$
एक चर समतल एक स्थिर बिन्दु (1, 2, 3) से गुजरता है। मूल बिन्दु से इस समतल पर डाले गए लम्ब के पाद का बिन्दुपथ है –
(A) $x^2 + y^2 + z^2 - x - 2y - 3z = 0$
(B) $x^2 + 2y^2 + 3z^2 - x - 2y - 3z = 0$
(C) $x^2 + 4y^2 + 9z^2 + x + 2y + 3 = 0$
(D) $x^2 + y^2 + z^2 + x + 2y + 3z = 0$

Sol. $OP \perp AP$
$\alpha(\alpha - 1) + \beta(\beta - 2) + \gamma(\gamma - 3) = 0$
$\therefore$ Locus of $P(\alpha, \beta, \gamma)$ is
$\therefore$ $P(\alpha, \beta, \gamma)$ का बिन्दुपथ है
$x^2 + y^2 + z^2 - x - 2y - 3z = 0$

Exercise-3

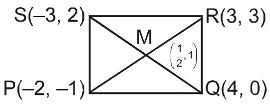
*** Marked Questions may have more than one correct option.**
*** चिन्हित प्रश्न एक से अधिक सही विकल्प वाले प्रश्न है -**

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

भाग - I : JEE (ADVANCED) / IIT-JEE (पिछले वर्षो) के प्रश्न

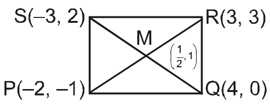
1. Let P, Q, R and S be the points on the plane with position vectors $-2\hat{i} - \hat{j}$, $4\hat{i}$, $3\hat{i} + 3\hat{j}$ and $-3\hat{i} + 2\hat{j}$ respectively. The quadrilateral PQRS must be a **[IIT-JEE-2010, Paper-1, (3, –1), 84]**
(A*) parallelogram, which is neither a rhombus nor a rectangle
(B) square
(C) rectangle, but not a square
(D) rhombus, but not a square
माना कि एक समतल में बिन्दु P, Q, R तथा S हैं। जिनके स्थिति सदिश क्रमशः $-2\hat{i} - \hat{j}$, $4\hat{i}$, $3\hat{i} + 3\hat{j}$ तथा $-3\hat{i} + 2\hat{j}$ हैं। तो चतुर्भुज PQRS होगा
(A) एक समानान्तर चतुर्भुज जो न समचतुर्भुज है और न ही एक आयत है
(B) एक वर्ग
(C) एक आयत परन्तु वर्ग नहीं
(D) एक समचतुर्भुज परन्तु वर्ग नहीं

Sol. PQ = $\sqrt{36+1} = \sqrt{37}$ = RS,
PS = $\sqrt{1+9} = \sqrt{10}$ = QR,
PQ $\neq$ PS
slope of PQ = $\frac{1}{6}$, slope of PS = – 3



PQ is not $\perp$ to PS
So it is parallelogram, which is neither a rhombus nor a rectangle

Hindi PQ = $\sqrt{36+1} = \sqrt{37}$ = RS,
PS = $\sqrt{1+9} = \sqrt{10}$ = QR,
PQ $\neq$ PS
PQ की प्रवणता = $\frac{1}{6}$, PS = – 3 की प्रवणता



PQ, PS के लम्बवत् नहीं है
इसलिए यह समान्तर चतुर्भुज जो न तो समचतुर्भुज और न ही आयत

2. If $\vec{a}$ and $\vec{b}$ are vectors in space given by $\vec{a} = \frac{\hat{i}-2\hat{j}}{\sqrt{5}}$ and $\vec{b} = \frac{2\hat{i}+\hat{j}+3\hat{k}}{\sqrt{14}}$, then the value of $\left(2\vec{a}+\vec{b}\right) \cdot \left[\left(\vec{a}\times\vec{b}\right)\times\left(\vec{a}-2\vec{b}\right)\right]$ is **[IIT-JEE-2010, Paper-1, (3, 0), 84]**

यदि तथा त्रिविम (space) में दो सदिश इस प्रकार है कि, $\vec{a} = \frac{\hat{i}-2\hat{j}}{\sqrt{5}}$ तथा $\vec{b} = \frac{2\hat{i}+\hat{j}+3\hat{k}}{\sqrt{14}}$ तब $\left(2\vec{a}+\vec{b}\right).\left[\left(\vec{a}\times\vec{b}\right)\times\left(\vec{a}-2\vec{b}\right)\right]$ का मान ज्ञात कीजिए।

Ans. 5

Sol. $\vec{a} = \frac{\hat{i}-2\hat{j}}{\sqrt{5}}$, $\vec{b} = \frac{2\hat{i}+\hat{j}+3\hat{k}}{\sqrt{14}}$

$|\vec{a}| = 1$, $|\vec{b}| = 1$

$\vec{a} \, . \, \vec{b} = 0$

$(2\vec{a}+\vec{b}) \, . \left\{ (\vec{a}\times\vec{b}) \times(\vec{a}-2\vec{b}) \right\}$

$= (2\vec{a}+\vec{b}).\left\{ (\vec{a} \, .(\vec{a}-2\vec{b})) \, \vec{b} - (\vec{b}.(\vec{a}-2\vec{b}) \,) \, \vec{a} \right\}$

$= (2\vec{a}+\vec{b}).\left\{ (\vec{a}. \, \vec{a} - 2\vec{a} \, . \, \vec{b}) \, \vec{b}-(\vec{b}. \, \vec{a}-2\vec{b}. \, \vec{b}) \, \vec{a} \right\}$

$= (2\vec{a}+\vec{b}).\left\{ (1-0) \, \vec{b}-(0-2) \, \vec{a} \right\}$

$= (2\vec{a}+\vec{b}).\left\{ \vec{b}+2\vec{a} \right\}$

$= 2 \, (\vec{a} \, . \, \vec{b})+4(\vec{a} \, . \, \vec{a})+\vec{b}. \, \vec{b}+2(\vec{b}. \, \vec{a}) = 0 + 4 + 1 + 0 = 5$

3. Two adjacent sides of a parallelogram ABCD are given by **[IIT-JEE-2010, Paper-2, (5, –2), 79]**
$\overrightarrow{AB} = 2\hat{i}+10\hat{j}+11\hat{k}$ and $\overrightarrow{AD} = -\hat{i}+2\hat{j}+2\hat{k}$. The side AD is rotated by an acute angle α in the plane of the parallelogram so that AD becomes AD′ . If AD′ makes a right angle with the side AB, then the cosine of the angle α is given by

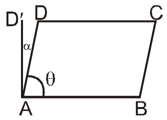
एक समानान्तर चतुर्भुज ABCD की दो संलग्न भुजाएँ निम्न प्रकार से दी गयी हैं।
$\overrightarrow{AB} = 2\hat{i}+10\hat{j}+11\hat{k}$ तथा $\overrightarrow{AD} = -\hat{i}+2\hat{j}+2\hat{k}$ समानान्तर चतुर्भुज के समतल में भुजा AD को एक न्यून कोण α से घुमाया जाता है जिससे कि AD, AD′ बन जाती है। यदि AD′ , AB के साथ समकोण बनाती है, तो cos (α) का मान निम्न है

(A) $\frac{8}{9}$ (B*) $\frac{\sqrt{17}}{9}$ (C) $\frac{1}{9}$ (D) $\frac{4\sqrt{5}}{9}$

Sol. $\cos\theta = \frac{-2+20+22}{15\times3} = \frac{8}{9}$ [Using dot product] [अदिश गुणन से]

$\theta + \alpha = 90º$

$\alpha = 90º - \theta$



$\cos\alpha = \sin\theta = \frac{\sqrt{17}}{9}$

4. Equation of the plane containing the straight line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$ and perpendicular to the plane containing the straight lines $\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$ and $\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$ is

सरल रेखा $\frac{x}{2}=\frac{y}{3}=\frac{z}{4}$ को अन्तर्विष्ट करने वाले तथा, रेखाओं $\frac{x}{3}=\frac{y}{4}=\frac{z}{2}$ एवं $\frac{x}{4}=\frac{y}{2}=\frac{z}{3}$ को अन्तर्विष्ट करने वाले (containing) समतल के लम्बवत् समतल का समीकरण निम्न है **[IIT-JEE-2010, Paper-1, (3, –1), 84]**

(A) $x + 2y - 2z = 0$ (B) $3x + 2y - 2z = 0$ (C*) $x - 2y + z = 0$ (D) $5x + 2y - 4z = 0$

Sol. Direction ratio's of normal to plane containing the straight line

रेखा को समाहित करने वाले समतल के अभिलम्ब के दिक्अनुपात

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 2 \\ 4 & 2 & 3 \end{vmatrix} = 8\hat{i} - \hat{j} - 10\hat{k}$$

Required planeअभीष्ट समतल $\begin{vmatrix} x-0 & y-0 & z-0 \\ 2 & 3 & 4 \\ 8 & -1 & -10 \end{vmatrix} = 0 \Rightarrow -26x + 52y - 26z = 0 \Rightarrow x - 2y + z = 0$

5. The number of 3 × 3 matrices A whose entries are either 0 or 1 and for which the system $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has exactly two distinct solutions, is

ऐसे 3 × 3 के आव्यूहों A की संख्या, जिनकी प्रविष्टियाँ 0 या 1 हैं तथा जिनके लिए निकाय $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ के ठीक (exactly) दो भिन्न हल हैं, निम्न है **[IIT-JEE-2010, Paper-1, (3, –1), 84]**

(A*) 0 (B) $2^9 - 1$ (C) 168 (D) 2

Sol. $a_1x + b_1y + c_1z = 1$

$a_2x + b_2y + c_2z = 0$

$a_3x + b_3y + c_3z = 0$

No three planes can meet at two distinct points. So number of matrices is 0

कोई भी तीन समतल किन्ही दो भिन्न बिन्दुओं पर नहीं मिल सकते इसलिए आव्यूहों की संख्या 0 है।

6. If the distance between the plane $Ax - 2y + z = d$ and the plane containing the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ is $\sqrt{6}$, then |d| is

यदि समतल $Ax - 2y + z = d$ तथा रेखाओं $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ तथा $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ को अन्तर्विष्ट करने वाले समतल के बीच की दूरी $\sqrt{6}$ हो, तो |d| का मान है। **[IIT-JEE-2010, Paper-1, (3, 0), 84]**

Ans. 6

Sol. Equation of plane is $\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$

$x - 2y + z = 0$ ....(1)

$Ax - 2y + z = d$ ....(2)

Compare $\frac{A}{1} = \frac{-2}{-2} = \frac{1}{1} \Rightarrow A = 1$

Distance between planes is $\left|\frac{d}{\sqrt{1+1+4}}\right| = \sqrt{6}$

$\Rightarrow \quad |d| = 6$

Hindi समतल का समीकरण $\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$

$x - 2y + z = 0$ ....(1)

$Ax - 2y + z = d$ ....(2)

तुलना करने पर $\frac{A}{1} = \frac{-2}{-2} = \frac{1}{1} \Rightarrow A = 1$

समतलों के मध्य दूरी $\left|\frac{d}{\sqrt{1+1+4}}\right| = \sqrt{6} \quad \Rightarrow \quad |d| = 6$

7. If the distance of the point P(1, –2, 1) from the plane $x + 2y - 2z = \alpha$, where $\alpha > 0$, is 5, then the foot of the perpendicular from P to the plane is

यदि बिन्दु P(1, –2, 1) से समतल $x + 2y - 2z = \alpha$, जहाँ $\alpha > 0$ की दूरी 5 है, तो बिन्दु P से समतल पर डाले गए लम्ब का पाद (foot) निम्न है **[IIT-JEE-2010, Paper-2, (5, –2), 79]**

(A*) $\left(\frac{8}{3}, \frac{4}{3}, -\frac{7}{3}\right)$ (B) $\left(\frac{4}{3}, -\frac{4}{3}, \frac{1}{3}\right)$ (C) $\left(\frac{1}{3}, \frac{2}{3}, \frac{10}{3}\right)$ (D) $\left(\frac{2}{3}, -\frac{1}{3}, \frac{5}{2}\right)$

Sol. $D = \left|\frac{1-4-2-\alpha}{3}\right| = 5$

$\alpha + 5 = 15 \qquad (\because \ \alpha > 0)$

$\Rightarrow \qquad \alpha = 10$

$\Rightarrow$ plane is $x + 2y - 2z - 10 = 0$

Let foot of perpendicular is (α, β, γ)

$$\frac{\alpha-1}{1} = \frac{\beta+2}{2} = \frac{\gamma-1}{-2} = -\left(\frac{1-4-2-10}{9}\right) = \frac{5}{3} \Rightarrow \alpha = \frac{8}{3}, \beta = \frac{4}{3}, \gamma = -\frac{7}{3}$$

Hindi $D = \left|\frac{1-4-2-\alpha}{3}\right| = 5$

$\alpha + 5 = 15 \qquad (\because \ \alpha > 0)$

$\Rightarrow \qquad \alpha = 10$

$\Rightarrow$ समतल $x + 2y - 2z - 10 = 0$

माना लम्बपाद (α, β, γ) है।

$$\frac{\alpha-1}{1} = \frac{\beta+2}{2} = \frac{\gamma-1}{-2} = -\left(\frac{1-4-2-10}{9}\right) = \frac{5}{3} \Rightarrow \alpha = \frac{8}{3}, \beta = \frac{4}{3}, \gamma = -\frac{7}{3}$$

8.✍ Match the statements in **Column-I** with those in **Column-II.** **[IIT-JEE-2010, Paper-2, (8, 0), 79]**

Column-I	**Column-II**
(A) A line from the origin meets the lines $\frac{x-2}{1} = \frac{y-1}{-2} = \frac{z+1}{1}$ and $\frac{x-\frac{8}{3}}{2} = \frac{y+3}{-1} = \frac{z-1}{1}$ at P and Q respectively. If length PQ = d, then d^2 is	(p) – 4
(B) The values of x satisfying $\tan^{-1}(x+3) - \tan^{-1}(x-3) = \sin^{-1}\left(\frac{3}{5}\right)$ are	(q) 0

(C) Non-zero vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ satisfy $\vec{a}.\vec{b} = 0$, $(\vec{b}-\vec{a}).(\vec{b}+\vec{c}) = 0$ and $2|\vec{b}+\vec{c}| = |\vec{b}-\vec{a}|$. If $\vec{a} = \mu\vec{b}+4\vec{c}$ then possible value of μ are — (r) 4

(D) Let f be the function on $[-\pi, \pi]$ given by — (s) 5

f(0) = 9 and f(x) = $\dfrac{\sin\left(\frac{9x}{2}\right)}{\sin\left(\frac{x}{2}\right)}$ for $x \neq 0$. The value — (t) 6

of $\dfrac{2}{\pi}\int_{-\pi}^{\pi} f(x)\ dx$ is

कॉलम **-I** में दिए गए वक्तव्यों का कॉलम **-II** में दिए वक्तव्यों से सुमेल करें।

कॉलम -I — **कॉलम -II**

(A) मूल बिन्दु से खींची गयी एक सरल रेखा, सरल रेखाओं $\dfrac{x-2}{1} = \dfrac{y-1}{-2} = \dfrac{z+1}{1}$ तथा $\dfrac{x-\frac{8}{3}}{2} = \dfrac{y+3}{-1} = \dfrac{z-1}{1}$ को क्रमशः P एवं Q पर काटती है। यदि लम्बाई PQ = d हो तो d^2 है– — (p) – 4

(B) $\tan^{-1}(x+3) - \tan^{-1}(x-3) = \sin^{-1}\left(\dfrac{3}{5}\right)$ को संतुष्ट करने वाले x के मान हैं– — (q) 0

(C) तीन अशुन्य सदिश $\vec{a}$, $\vec{b}$ तथा $\vec{c}$ समीकरण $\vec{a}.\vec{b} = 0$, $(\vec{b}-\vec{a}).(\vec{b}+\vec{c}) = 0$ तथा $2|\vec{b}+\vec{c}| = |\vec{b}-\vec{a}|$ को संतुष्ट करते हैं। यदि $\vec{a} = \mu\vec{b}+4\vec{c}$ हो, तो μ के सम्भव मान है– — (r) 4

(D) मानाकि $[-\pi, \pi]$ पर फलन f ; f(0) = 9 तथा f(x) = $\dfrac{\sin\left(\frac{9x}{2}\right)}{\sin\left(\frac{x}{2}\right)}$ — (s) 5

for $x \neq 0$. The value of $\dfrac{2}{\pi}\int_{-\pi}^{\pi} f(x)\ dx$ का मान है– — (t) 6

Ans. (A) → (t), (B) → (p, r), (C) → (q) (JEE given q, s) (D) → (r)

Sol. **(A)** Let the line through origin is $\dfrac{x}{\lambda} = \dfrac{y}{\mu} = \dfrac{z}{1}$

$\Rightarrow x = \lambda z,\ y = \mu z$ ....(1)

To find point of intersection of line (1) and line $\dfrac{x-2}{1} = \dfrac{y-1}{-2} = \dfrac{z+1}{1}$ ....(2)

we have $\dfrac{\lambda z-2}{1} = \dfrac{\mu z-1}{-2} = z + 1$

$\Rightarrow z = \dfrac{3}{\lambda-1} =$

$\Rightarrow \lambda + 3\mu + 5 = 0$ ....(3)

To find point of intersection of line (1) and line = = ...(4)

we have $\dfrac{\lambda z - \frac{8}{3}}{2} = \dfrac{\mu z + 3}{-1} = \dfrac{z-1}{1}$

$\Rightarrow z = \dfrac{2}{3(\lambda - 2)} = \dfrac{-2}{\mu + 1}$

$\Rightarrow 3\lambda + \mu = 5$ ....(5)

Solving (3) and (5), $\lambda = \dfrac{5}{2}$ and $\mu = -\dfrac{5}{2}$

$\therefore$ z = 2, x = 5, y = – 5 for point P

and $z = \dfrac{4}{3}$, $x = \dfrac{10}{3}$, $y = -\dfrac{10}{3}$ for point Q

$\therefore PQ^2 = \dfrac{4}{9} + \dfrac{25}{9} + \dfrac{25}{9} = 6$

(B) $\tan^{-1}(x+3) - \tan^{-1}(x-3) = \sin^{-1}(3/5)$

$\Rightarrow \tan^{-1}\left(\dfrac{x+3-x+3}{1+x^2-9}\right) = \tan^{-1}(3/4)$

$\Rightarrow \dfrac{6}{x^2-8} = \dfrac{3}{4} \Rightarrow x^2 = 16$

$\therefore x = \pm 4$

(C) Since . $\vec{a}\,\vec{b} = 0$

$\therefore$ Let $\vec{b} = \lambda_1 \hat{i}$, $\vec{a} = \lambda_2 \hat{j}$

Now $2|\vec{b} + \vec{c}\,| = |\vec{b} - \vec{a}\,|$ & $\vec{a} = \mu\vec{b} + 4\vec{c} \Rightarrow 2\left|\lambda_1\hat{i} + \dfrac{\lambda_2\hat{j} - \lambda_1 \quad \mu\vec{b}}{4}\right| = |\lambda_1\hat{i} - \lambda_2\hat{j}|$

$|\lambda_1(4-\mu) \quad \hat{i} + \lambda_2\hat{j}| = 2|\lambda_1 \quad \hat{i} + \lambda_2\hat{j}|$

squaring

$\lambda_1^{\,2}(4-\mu)^2 + \lambda_2^{\,2} = 4\lambda_1^{\,2} + 4\lambda_2^{\,2} \quad \Rightarrow \quad 3\lambda_2^{\,2} = (12 + \mu^2 - 8\mu)\ \lambda_1^{\,2}$ ....(1)

Also $(\vec{b} - \vec{a}\,).(\vec{b} + \vec{c}\,) = 0$

$\Rightarrow \quad (\lambda_1\hat{i} - \lambda_2\hat{j}) \,.\, \left(\lambda_1\hat{i} + \dfrac{\lambda_2\hat{j} - \lambda_1 \quad \mu\hat{i}}{4}\right) = 0 \quad \Rightarrow \quad \dfrac{\lambda_1^{\,2}(4-\mu) - \lambda_2^{\,2}}{4} = 0$

$\Rightarrow \quad \lambda_2^{\,2} = \lambda_1^{\,2}(4-\mu)$ ....(2)

from (1) & (2)

$12 + \mu^2 - 8\mu = 12 - 3\mu$

$\Rightarrow \quad \mu^2 - 5\mu = 0 \quad \Rightarrow \quad \mu = 0, 5$ ($\mu = 5$ reject)

(D) $I = \dfrac{2}{\pi}\displaystyle\int_{-\pi}^{\pi} \dfrac{\sin\frac{9x}{2}}{\sin\frac{x}{2}}\,dx = \dfrac{4}{\pi}\int_{0}^{\pi} \dfrac{\sin\frac{9x}{2}\cos\frac{x}{2}}{\sin\frac{x}{2}\cos\frac{x}{2}}\,dx = \dfrac{8}{\pi}\int_{0}^{\pi} \dfrac{\sin\frac{9x}{2}\cos\frac{x}{2}}{\sin x}\,dx$

$I = \dfrac{4}{\pi}\displaystyle\int_{0}^{\pi} \dfrac{\sin 5x + \sin 4x}{\sin x}\,dx$ ......(i)

(using $\displaystyle\int_{0}^{b} f(x)dx = \int_{0}^{b} f(a+b-x)dx$)

$$= \frac{4}{\pi}\int_0^{\pi} \frac{\sin 5x - \sin 4x}{\sin x}\, dx \qquad ......(ii)$$

Add (i) and (ii)

$$I = \frac{4}{\pi}\int_0^{\pi} \frac{\sin 5x}{\sin x}\, dx$$

Consider

$$I_k - I_{k-2} = \frac{4}{\pi}\int_0^{\pi} \frac{\sin kx - \sin(k-2)\ x}{\sin x}\, dx = \frac{8}{\pi}\int_0^{\pi} \frac{\cos(k-1)x \sin x}{\sin x}\, dx$$

$I_k = I_{k-2}$

so $I_5 = I_3 \qquad \Rightarrow \qquad I_5 = I_1 = \frac{4}{\pi}\int_0^{\pi} dx = 4$

Aliter

Let $I = \frac{2}{\pi}\int_{-\pi}^{\pi} \frac{\sin\ (9x/2)}{\sin(x/2)}\, dx$

$$I = \frac{4}{\pi}\int_0^{\pi} \frac{\sin(9x/2)}{\sin(x/2)}\, dx \quad ....(1) \qquad (\because\ f(x) \text{ is even function})$$

$$I = \frac{4}{\pi}\int_0^{\pi} \frac{\cos(9x/2)}{\cos(x/2)}\, dx \quad ....(2)$$

(using $\int_0^b f(x)dx = \int_0^b f(a+b-x)dx$)

Add (1) & (2)

$$I = \frac{4}{\pi}\int_0^{\pi} \frac{\sin 5x}{2\sin(x/2)\cos(x/2)}\, dx = \frac{4}{\pi}\int_0^{\pi} \frac{\sin 5x}{\sin x}\, dx$$

$$\Rightarrow \quad I = \frac{8}{\pi}\int_0^{\pi/2} \frac{\sin 5x}{\sin x}\, dx$$

$$\Rightarrow \quad I = \frac{8}{\pi}\int_0^{\pi/2} \left(\frac{16\sin^5 x - 20\sin^3 x + 5\sin x}{\sin x}\right) dx$$

$$\Rightarrow \quad I = \frac{8}{\pi}\int_0^{\pi/2} (16\sin^4 x - 20\sin^2 x + 5\ )\ dx$$

$$\Rightarrow \quad I = \frac{8}{\pi}\left[16x\ \frac{3\times 1\times \pi}{4\times 2\times 2} - 20\times\frac{1\times\pi}{2\times 2} + \frac{5\pi}{2}\right]$$

$$\Rightarrow \quad I = \frac{8}{\pi}\left[3\pi - 5\pi + \frac{5\pi}{2}\right]$$

$\Rightarrow \quad I = 4$

Hindi (A) माना मूल बिन्दु से जाने वाली रेखा $\frac{x}{\lambda} = \frac{y}{\mu} = \frac{z}{1}$

$\Rightarrow\ x = \lambda z\ ,\ y = \mu z \qquad ....(1)$

रेखा (1) और रेखा $\frac{x-2}{1} = \frac{y-1}{-2} = \frac{z+1}{1}$ के प्रतिच्छेदन बिन्दु के लिए ....(2)

$$\frac{\lambda z - 2}{1} = \frac{\mu z - 1}{-2} = z + 1$$

$\Rightarrow z = \dfrac{3}{\lambda - 1} =$

$\Rightarrow \lambda + 3\mu + 5 = 0$ ....(3)

रेखा (1) और रेखा $\dfrac{x - \frac{8}{3}}{2} = \dfrac{y+3}{-1} = \dfrac{z-1}{1}$ के प्रतिच्छेदन के लिए ....(4)

$\dfrac{\lambda z - \frac{8}{3}}{2} = \dfrac{\mu z + 3}{-1} = \dfrac{z-1}{1}$

$\Rightarrow z = \dfrac{2}{3(\lambda - 2)} = \dfrac{-2}{\mu + 1}$

$\Rightarrow 3\lambda + \mu = 5$ ....(5)

(3) और (5) से , $\lambda = \dfrac{5}{2}$ और $\mu = -\dfrac{5}{2}$

$\therefore$ बिन्दु P के लिए $z = 2, x = 5, y = -5$

और $z = \dfrac{4}{3}$, $x = \dfrac{10}{3}$, $y = -\dfrac{10}{3}$

$\therefore$ और बिन्दु Q के लिए $PQ^2 = \dfrac{4}{9} + \dfrac{25}{9} + \dfrac{25}{9} = 6$

(B) $\tan^{-1}(x + 3) - \tan^{-1}(x - 3) = \sin^{-1}(3/5)$

$\Rightarrow \tan^{-1}\left(\dfrac{x+3-x+3}{1+x^2-9}\right) = \tan^{-1}(3/4)$

$\Rightarrow \dfrac{6}{x^2 - 8} = \dfrac{3}{4} \Rightarrow x^2 = 16$

$\therefore x = \pm 4$

(C) चूंकि $\vec{a}\,\vec{b} = 0$

$\therefore$ माना $\vec{b} = \lambda_1 \hat{i}$, $\vec{a} = \lambda_2 \hat{j}$

अब $2|\vec{b} + \vec{c}\,| = |\vec{b} - \vec{a}\,|$ और $\vec{a} = \mu \vec{b} + 4\vec{c}$ $\Rightarrow 2\left|\lambda_1 \hat{i} + \dfrac{\lambda_2 \hat{j} - \lambda_1 \ \mu \vec{b}}{4}\right| = |\lambda_1 \hat{i} - \lambda_2 \hat{j}\,|$

$|\lambda_1(4-\mu)\ \hat{i} + \lambda_2 \hat{j}\,| = 2|\lambda_1\ \hat{i} + \lambda_2 \hat{j}\,|$

वर्ग करने पर

$\lambda_1^{\,2}(4-\mu)^2 + \lambda_2^{\,2} = 4\lambda_1^{\,2} + 4\lambda_2^{\,2} \Rightarrow 3\lambda_2^{\,2} = (12 + \mu^2 - 8\mu)\ \lambda_1^{\,2}$ ....(1)

तथा $(\vec{b} - \vec{a}\,).(\vec{b} + \vec{c}\,) = 0$

$\Rightarrow (\lambda_1 \hat{i} - \lambda_2 \hat{j}) . \left(\lambda_1 \hat{i} + \dfrac{\lambda_2 \hat{j} - \lambda_1\ \mu \hat{i}}{4}\right) = 0 \Rightarrow \dfrac{\lambda_1^{\,2}(4-\mu) - \lambda_2^{\,2}}{4} = 0$

$\Rightarrow \lambda_2^{\,2} = \lambda_1^{\,2}(4 - \mu)$ ....(2)

(1) व (2) से

$12 + \mu^2 - 8\mu = 12 - 3\mu$

$\Rightarrow \mu^2 - 5\mu = 0 \Rightarrow \mu = 0, 5$ ($\mu = 5$ reject)

(D) $I = \dfrac{2}{\pi}\displaystyle\int_{-\pi}^{\pi} \dfrac{\sin\frac{9x}{2}}{\sin\frac{x}{2}}\,dx = \dfrac{4}{\pi}\displaystyle\int_{0}^{\pi} \dfrac{\sin\frac{9x}{2}\cos\frac{x}{2}}{\sin\frac{x}{2}\cos\frac{x}{2}}\,dx = \dfrac{8}{\pi}\displaystyle\int_{0}^{\pi} \dfrac{\sin\frac{9x}{2}\cos\frac{x}{2}}{\sin x}\,dx$

$$I = \frac{4}{\pi}\int_0^{\pi} \frac{\sin 5x + \sin 4x}{\sin x}\, dx \qquad ......(i)$$

$\left(\int_0^b f(x)dx = \int_0^b f(a+b-x)dx\right.$ के उपयोग से)

$$= \frac{4}{\pi}\int_0^{\pi} \frac{\sin 5x - \sin 4x}{\sin x}\, dx \qquad ......(ii)$$

(i) और (ii) को जोड़ने पर

$$I = \frac{4}{\pi}\int_0^{\pi} \frac{\sin 5x}{\sin x}\, dx$$

मानाकि

$$I_k - I_{k-2} = \frac{4}{\pi}\int_0^{\pi} \frac{\sin kx - \sin(k-2)\ x}{\sin x}\, dx = \frac{8}{\pi}\int_0^{\pi} \frac{\cos(k-1)x \sin x}{\sin x}\, dx$$

$$I_k = I_{k-2}$$

अतः $I_5 = I_3 \quad \Rightarrow \quad I_5 = I_1 = \frac{4}{\pi}\int_0^{\pi} dx = 4$

वैकल्पिक

माना $\quad I = \dfrac{2}{\pi}\displaystyle\int_{-\pi}^{\pi} \frac{\sin\ (9x/2)}{\sin(x/2)}\, dx$

$$I = \frac{4}{\pi}\int_0^{\pi} \frac{\sin(9x/2)}{\sin(x/2)}\, dx \qquad ....(1) \qquad (\because\ f(x) \text{ सम फलन है})$$

$$I = \frac{4}{\pi}\int_0^{\pi} \frac{\cos(9x/2)}{\cos(x/2)}\, dx \qquad ....(2)$$

$\left(\int_0^b f(x)dx = \int_0^b f(a+b-x)dx\right.$ के उपयोग से)

(1) व (2) को जोड़ने पर

$$I = \frac{4}{\pi}\int_0^{\pi} \frac{\sin 5x}{2\sin(x/2)\cos(x/2)}\, dx = \frac{4}{\pi}\int_0^{\pi} \frac{\sin 5x}{\sin x}\, dx$$

$$\Rightarrow \quad I = \frac{8}{\pi}\int_0^{\pi/2} \frac{\sin 5x}{\sin x}\, dx$$

$$\Rightarrow \quad I = \frac{8}{\pi}\int_0^{\pi/2} \left(\frac{16\sin^5 x - 20\sin^3 x + 5\sin x}{\sin x}\right) dx$$

$$\Rightarrow \quad I = \frac{8}{\pi}\int_0^{\pi/2} (16\sin^4 x - 20\sin^2 x + 5\ )\ dx$$

$$\Rightarrow \quad I = \frac{8}{\pi}\left[16x\ \frac{3\times 1\times \pi}{4\times 2\times 2} - 20\times\frac{1\times\pi}{2\times 2} + \frac{5\pi}{2}\right]$$

$$\Rightarrow \quad I = \frac{8}{\pi}\left[3\pi - 5\pi + \frac{5\pi}{2}\right]$$

$$\Rightarrow \quad I = 4$$

9. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - \hat{j} - \hat{k}$ be three vectors. A vector $\vec{v}$ in the plane of $\vec{a}$ and $\vec{b}$, whose projection on $\vec{c}$ is $\frac{1}{\sqrt{3}}$, is given by

माना कि $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$ और $\vec{c} = \hat{i} - \hat{j} - \hat{k}$ तीन सदिश है। एक सदिश $\vec{v}$ जो $\vec{a}$ और $\vec{b}$ समतल में स्थित है, और जिसका $\vec{c}$ पर प्रक्षेप $\frac{1}{\sqrt{3}}$ है, निम्न है– **[IIT-JEE 2011, Paper-1, (3, –1), 80]**

(A) $\hat{i} - 3\hat{j} + 3\hat{k}$ (B) $-3\hat{i} - 3\hat{j} - \hat{k}$ (C*) $3\hat{i} - \hat{j} + 3\hat{k}$ (D) $\hat{i} + 3\hat{j} - 3\hat{k}$

Ans. **(C)**

Sol. Let $\vec{v} = \lambda\vec{a} + \mu\vec{b}$

$\Rightarrow \vec{v} = (\lambda+\mu)\ \hat{i} + (\lambda-\mu) + (\lambda+\mu)\hat{k}$

Now $\vec{v}.\ \hat{c} = \frac{1}{\sqrt{3}}$

$\Rightarrow \frac{(\lambda+\mu)-(\lambda-\mu)-(\lambda+\mu)}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

$\Rightarrow \mu - \lambda = 1$

$\mu = \lambda + 1$

$\therefore \vec{v} = (2\lambda+1)\, i - \hat{j} + (2\lambda+1)\ \hat{k}$

For $\lambda = 1$, $\vec{v} = 3\hat{i} - \hat{j} + 3\hat{k}$

Hindi माना $\vec{v} = \lambda\vec{a} + \mu\vec{b}$

$\Rightarrow \vec{v} = (\lambda+\mu)\ \hat{i} + (\lambda-\mu) + (\lambda+\mu)\hat{k}$

अब $\vec{v}.\ \hat{c} = \frac{1}{\sqrt{3}}$

$\Rightarrow \frac{(\lambda+\mu)-(\lambda-\mu)-(\lambda+\mu)}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

$\Rightarrow \mu - \lambda = 1$

$\mu = \lambda + 1$

$\therefore \vec{v} = (2\lambda+1)\, i - \hat{j} + (2\lambda+1)\ \hat{k}$

$\lambda = 1$, $\vec{v} = 3\hat{i} - \hat{j} + 3\hat{k}$ के लिए

10*. The vector(s) which is/are coplanar with vectors $\hat{i} + \hat{j} + 2\hat{k}$ and $\hat{i} + 2\hat{j} + \hat{k}$, and perpendicular to the vector $\hat{i} + \hat{j} + \hat{k}$ is/are

वह (वे) सदिश जो सदिशों $\hat{i} + \hat{j} + 2\hat{k}$ तथा $\hat{i} + 2\hat{j} + \hat{k}$ द्वारा बने तल में स्थित है(हैं), और सदिश $\hat{i} + \hat{j} + \hat{k}$ के लम्बवत है(हैं), निम्न है(हैं) **[IIT-JEE 2011, Paper-1, (4, 0), 80]**

(A*) $\hat{j} - \hat{k}$ (B) $-\hat{i} + \hat{j}$ (C) $\hat{i} - \hat{j}$ (D*) $-\hat{j} + \hat{k}$

Ans. **(A, D)**

Sol. $\bar{a} = \hat{i} + \hat{j} + 2\hat{k}$

$\bar{b} = \hat{i} + 2\hat{j} + \hat{k}$

$\bar{c} = \hat{i} + \hat{j} + \hat{k}$

Required vector is $\lambda\bar{c} \times (\bar{a} \times \bar{b})$

$\lambda[(\bar{c}.\bar{b})\bar{a} - (\bar{c}.\bar{a})\bar{b}]$

$\lambda[(1+2+1) - (\hat{i} + \hat{j} + 2\hat{k}) - (1+1+2)(\hat{i} + 2\hat{j} + \hat{k})]$

$\lambda[-4\hat{j} + 4\hat{k}]$

so our vector in parallel $-\hat{j} + \hat{k}$

Hindi $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$

$\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$

$\vec{c} = \hat{i} + \hat{j} + \hat{k}$

अभीष्ट सदिश $\lambda\vec{c} \times (\vec{a} \times \vec{b})$ है

$\lambda[(\vec{c}.\vec{b})\vec{a} - (\vec{c}.\vec{a})\vec{b}]$

$\lambda[(1+2+1) - (\hat{i}+\hat{j}+2\hat{k}) - (1+1+2)(\hat{i}+2\hat{j}+\hat{k})]$

$\lambda[-4\hat{j} + 4\hat{k}]$

सदिश $-\hat{j} + \hat{k}$ के समान्तर है।

11. Let $\vec{a} = -\hat{i} - \hat{k}$, $\vec{b} = -\hat{i} + \hat{j}$ and $\vec{c} = \hat{i} + 2\hat{j} + 3\hat{k}$ be three given vectors. If $\vec{r}$ is a vector such that $\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$ and $\vec{r} \cdot \vec{a} = 0$, then the value of $\vec{r} \cdot \vec{b}$ is **[IIT-JEE 2011, Paper-2, (4, 0), 80]**

माना कि तीन सदिश $\vec{a} = -\hat{i} - \hat{k}$, $\vec{b} = -\hat{i} + \hat{j}$ और $\vec{c} = \hat{i} + 2\hat{j} + 3\hat{k}$ दिये गये है। यदि एक सदिश $\vec{r}$ इस प्रकार का है कि $\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$ और $\vec{r} \cdot \vec{a} = 0$ मान्य हैं, तब $\vec{r} \cdot \vec{b}$ का मान है।

Ans. 9

Sol. $(\vec{r} - \vec{c}) \times \vec{b} = 0$

$\vec{r} - \vec{c} = \lambda\vec{b} \Rightarrow \vec{r} = \vec{c} + \lambda\vec{b} \quad \lambda \in R$

$\because \vec{r} \cdot \vec{a} = 0$

$\Rightarrow (\vec{c} + \lambda\vec{b}) \cdot \vec{a} = 0$

$\Rightarrow ((\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(-\hat{i} + \hat{j})) \cdot (-\hat{i} - \hat{k}) = 0$

$\Rightarrow ((1-\lambda)\hat{i} + (2+\lambda)\hat{j} + 3\hat{k}) \cdot (-\hat{i} - \hat{k}) = 0$

$\Rightarrow \lambda - 1 - 3 = 0$

$\Rightarrow \lambda = 4$

so अतः $\vec{r} \cdot \vec{b} = (-3\hat{i} + 6\hat{j} + 3\hat{k}) \cdot (-\hat{i} + \hat{k})$

$= 3 + 6 = 9$

12. Match the statements given in **Column-I** with the values given in **Column-II**

Column-I		**Column-II**	
(A)	If $\vec{a} = \hat{j} + \sqrt{3}\,\hat{k}$, $\vec{b} = -\hat{j} + \sqrt{3}\,\hat{k}$ and $\vec{c} = 2\sqrt{3}\,\hat{k}$ form a triangle, then the internal angle of the triangle between $\vec{a}$ and $\vec{b}$ is	(p)	$\frac{\pi}{6}$
(B)	If $\int_a^b (f(x) - 3x)\,dx = a^2 - b^2$, then the value of $f\left(\frac{\pi}{6}\right)$ is	(q)	$\frac{2\pi}{3}$
(C)	The value of $\frac{\pi^2}{\ln 3}\int_{7/6}^{5/6} \sec(\pi x)\,dx$ is	(r)	$\frac{\pi}{3}$
(D)	The maximum value of $\left|\text{Arg}\left(\frac{1}{1-z}\right)\right|$ for $\|z\| = 1$, $z \neq 1$ is given by	(s)	π
		(t)	$\frac{\pi}{2}$

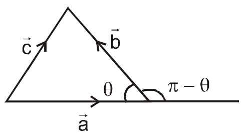
कॉलम I में दिये गये वक्तव्यों का कॉलम II में दिये गये मानों से सुमेल करें **[IIT-JEE 2011, Paper-2, (8, 0), 80]**

कॉलम-I **कॉलम-II**

(A)	यदि $\vec{a} = \hat{j} + \sqrt{3}\ \hat{k}$, $\vec{b} = -\hat{j} + \sqrt{3}\ \hat{k}$ और $\vec{c} = 2\sqrt{3}\ \hat{k}$ एक त्रिभुज निर्मित करते है, तो त्रिभुज का आन्तरिक कोण जो $\vec{a}$ और $\vec{b}$ के बीच स्थित है	(p)	$\frac{\pi}{6}$
(B)	यदि $\int_a^b (f(x) - 3x)\ dx = a^2 - b^2$, तो $f\left(\frac{\pi}{6}\right)$ का मान है	(q)	$\frac{2\pi}{3}$
(C)	$\frac{\pi^2}{\ln 3}\int_{7/6}^{5/6} \sec\ (\pi x)\ dx$ का मान है	(r)	$\frac{\pi}{3}$
(D)	यदि $\lvert z\rvert = 1$ और $z \neq 1$ तो $\left\lvert \text{Arg}\left(\frac{1}{1-z}\right)\right\rvert$ का उच्चतम मान है	(s)	π
		(t)	$\frac{\pi}{2}$

Ans. **(A) → (q), (B) → (p), (C) → (s), (D) → (t)**

Sol. **(A)** $\cos(\pi - \theta) = \frac{\vec{a}\ .\ \vec{b}}{|\vec{a}\ |\ |\vec{b}|} = \frac{-1+3}{\sqrt{1+3}\ \ \sqrt{1+3}} = \frac{2}{4}$



$-\cos\theta = \frac{1}{2}$

$\theta = \frac{2\pi}{3}$

(B) Using Leibeintz Theorem

$f(b) - 3b = -2b$

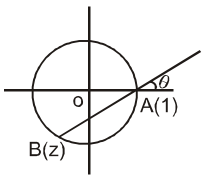
$f(b) = b$

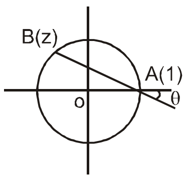
(C) $\frac{\pi^2}{\ell n 3}\int_{7/6}^{5/6} (\sec \pi x)\ \ dx = \frac{\pi^2}{\ell n 3}\left\{\frac{\ell n\,|\sec \pi x + \tan \pi x\,|}{\pi}\right\}_{7/6}^{5/6}$

$$= \frac{\pi^2}{\ell n 3}\left\{\frac{\ell n\ \left|\left(-\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right)\right| - \ell n\ \left|-\frac{2}{3} + \frac{1}{\sqrt{3}}\right|}{\pi}\right\} = \frac{\pi^2}{\ell n 3}\left\{\frac{\ell n\left|\frac{3}{\sqrt{3}} \times \frac{\sqrt{3}}{1}\right|}{\pi}\right\} = \pi$$

(D) $z\ (z \neq 1)$ lies on circle with center 0, radius 1

$\text{Arg}\left(\frac{1}{1-z}\right) = \text{Arg } 1 - \text{Arg } (1 - z) =$ angle between OA and BA



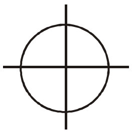


Absolute value of angle between OA and BA tends to $\frac{\pi}{2}$ as B tends to A.

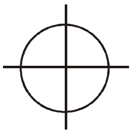
Alter # 1

$$\left|\arg\left(\frac{1}{1-z}\right)\right| = |\arg 1 - \arg(1-z)| = |\arg(1-z)|$$

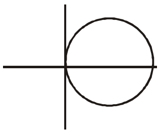
as $|z| = 1$ i.e. z lies on circle



$\Rightarrow$ $-z$ lies on circle



$\Rightarrow$ $1 - z$ lies on circle



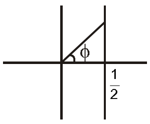
$\Rightarrow$ $\max|\arg(1-z)| = \frac{\pi}{2}$

Alter # 2

$z = e^{i\theta}$

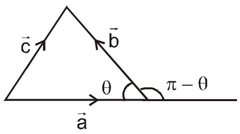
$$\frac{1}{1-z} = \frac{1}{2\sin^2\frac{\theta}{2} - i\sin\theta} = \frac{1}{2\sin\frac{\theta}{2}}\left(\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}\right) = \frac{1}{2} + i\frac{1}{2}\cot\frac{\theta}{2}$$

Locus is $\frac{1}{1-z}$ is $x = \frac{1}{2}$



Maximum value of ϕ tends to $\frac{\pi}{2}$

Hindi **(A)** $\cos(\pi - \theta) = \frac{\vec{a}\ .\ \vec{b}}{|\vec{a}|\ |\vec{b}|} = \frac{-1+3}{\sqrt{1+3}\ \sqrt{1+3}} = \frac{2}{4}$



$-\cos\theta = \frac{1}{2}$

$\theta = \frac{2\pi}{3}$

(B) Leibeintz Theorem से

$f(b) - 3b = -2b$

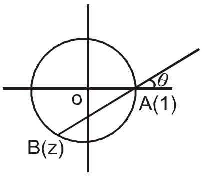
$f(b) = b$

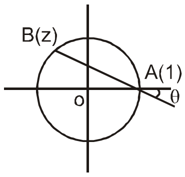
(C) $\dfrac{\pi^2}{\ell n 3}\displaystyle\int_{7/6}^{5/6}(\sec \pi x)\ dx = \dfrac{\pi^2}{\ell n 3}\left\{\dfrac{\ell n\,|\sec \pi x + \tan \pi x\,|}{\pi}\right\}_{7/6}^{5/6}$

$$= \frac{\pi^2}{\ell n 3}\left\{\frac{\ell n\ \left|\left(-\frac{2}{\sqrt{3}}-\frac{1}{\sqrt{3}}\right)\right| - \ell n\ \left|-\frac{2}{3}+\frac{1}{\sqrt{3}}\right|}{\pi}\right\} = \frac{\pi^2}{\ell n 3}\left\{\frac{\ell n\left|\frac{3}{\sqrt{3}}\times\frac{\sqrt{3}}{1}\right|}{\pi}\right\} = \pi$$

(D) $z\ (z \neq 1)$ वृत्त पर स्थित होंगे जिसका केन्द्र 0 और त्रिज्या 1 है।

Arg $\left(\dfrac{1}{1-z}\right)$ = Arg 1 – Arg (1 – z) = OA और BA का मध्य का कोण



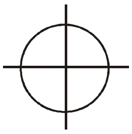


OA और BA के मध्य के कोण का निरपेक्ष मान $\dfrac{\pi}{2}$ की ओर अग्रसर है जबकि B, A की ओर अग्रसर है।

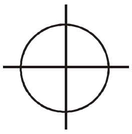
Alter # 1

$\left|\arg\left(\dfrac{1}{1-z}\right)\right|$ = |arg 1 – arg (1 – z)| = |arg (1 – z)|

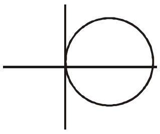
चूंकि |z| = 1 i.e. z वृत्त पर स्थित है।



$\Rightarrow$ – z वृत्त पर स्थित है।



$\Rightarrow$ 1 – z वृत्त पर स्थित है।



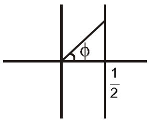
$\Rightarrow$ max |arg (1 – z)| = $\dfrac{\pi}{2}$

Alter # 2

$z = e^{i\theta}$

$$\frac{1}{1-z} = \frac{1}{2\sin^2\frac{\theta}{2} - i\sin\theta} = \frac{1}{2\sin\frac{\theta}{2}}\left(\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}\right) = \frac{1}{2} + i\frac{1}{2}\ \cot\frac{\theta}{2}$$

बिन्दुपथ $\dfrac{1}{1-z}$, $x = \dfrac{1}{2}$



ϕ का अधिकतम मान $\frac{\pi}{2}$ की ओर है।

13. The point P is the intersection of the straight line joining the points Q(2,3,5) and R(1, –1, 4) with the plane
$5x - 4y - z = 1$. If S is the foot of the perpendicular drawn from the point T(2, 1,4) to QR, then the length of the line segment PS is **[IIT-JEE 2012, Paper-1, (3, –1), 70]**

बिन्दु P बिंदुओं Q(2, 3, 5) और R(1, –1, 4) से गुजरने वाली सरल रेखा एवं समतल $5x - 4y - z = 1$ का प्रतिच्छेदी बिंदु है। यदि बिंदु T(2, 1, 4) से QR पर डाले गये लम्ब का लम्ब–पाद S है तो रेखा–खण्ड PS की लम्बाई निम्न है–

(A*) $\frac{1}{\sqrt{2}}$ (B) $\sqrt{2}$ (C) 2 (D) $2\sqrt{2}$

Sol. Ans. (A)

Equation of QR is

$$\frac{x-2}{1} = \frac{y-3}{4} = \frac{z-5}{1}$$

Let $P \equiv (2 + \lambda, 3 + 4\lambda, 5 + \lambda)$

$$10 + 5\lambda - 12 - 16\lambda - 5 - \lambda = 1$$
$$-7 - 12\lambda = 1$$
$$\Rightarrow \quad \lambda = \frac{-2}{3}$$

then $P \equiv \left(\frac{4}{3}, \frac{1}{3}, \frac{13}{3}\right)$

Let $S = (2 + \mu, 3 + 4\mu, 5 + \mu)$

$$\vec{TS} = (\mu)\hat{i} + (4\mu + 2)\hat{j} + (\mu + 1)\hat{k}$$
$$\vec{TS} \,.\, (\hat{i} + 4\hat{j} + \hat{k}) = 0$$
$$\mu + 16\mu + 8 + \mu + 1 = 0$$
$$\mu = -\frac{1}{2}$$
$$S = \left(\frac{3}{2}, 1, \frac{9}{2}\right)$$
$$PS = \sqrt{\left(\frac{4}{3} - \frac{3}{2}\right)^2 + \frac{4}{9} + \left(\frac{13}{3} - \frac{9}{2}\right)^2} = \sqrt{\frac{1}{36} + \frac{4}{9} + \frac{1}{36}} = \sqrt{\frac{1}{18} + \frac{4}{9}} = \sqrt{\frac{9}{18}} = \frac{1}{\sqrt{2}}$$

Hindi Ans. (A)

QR का समीकरण होगा

$$\frac{x-2}{1} = \frac{y-3}{4} = \frac{z-5}{1}$$

माना $P \equiv (2 + \lambda, 3 + 4\lambda, 5 + \lambda)$

$$10 + 5\lambda - 12 - 16\lambda - 5 - \lambda = 1$$
$$-7 - 12\lambda = 1$$
$$\Rightarrow \quad \lambda = \frac{-2}{3}$$

तो $P \equiv \left(\frac{4}{3}, \frac{1}{3}, \frac{13}{3}\right)$

माना $S = (2 + \mu, 3 + 4\mu, 5 + \mu)$

$\vec{TS} = (\mu)\hat{i} + (4\mu+2)\hat{j} + (\mu+1)\hat{k}$

$\vec{TS} \cdot (\hat{i} + 4\hat{j} + \hat{k}) = 0$

$\mu + 16\mu + 8 + \mu + 1 = 0$

$\mu = -\frac{1}{2}$

$S = \left(\frac{3}{2}, 1, \frac{9}{2}\right)$

$PS = \sqrt{\left(\frac{4}{3}-\frac{3}{2}\right)^2 + \frac{4}{9} + \left(\frac{13}{3}-\frac{9}{2}\right)^2}$

$= \sqrt{\frac{1}{36} + \frac{4}{9} + \frac{1}{36}}$

$= \sqrt{\frac{1}{18} + \frac{4}{9}} = \sqrt{\frac{9}{18}} = \frac{1}{\sqrt{2}}$

14. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors satisfying $|\vec{a} - \vec{b}|^2 + |\vec{b} - \vec{c}|^2 + |\vec{c} - \vec{a}|^2 = 9$, then $|2\vec{a} + 5\vec{b} + 5\vec{c}|$ is **[IIT-JEE 2012, Paper-1, (4, 0), 70]**

यदि $\vec{a}$, $\vec{b}$ और $\vec{c}$ इकाई सदिश (unit vectors) हैं जो $|\vec{a} - \vec{b}|^2 + |\vec{b} - \vec{c}|^2 + |\vec{c} - \vec{a}|^2 = 9$ को संतुष्ट करते हैं तब $|2\vec{a} + 5\vec{b} + 5\vec{c}|$ का मान है।

Sol. **Ans 3**

$6 - 2\vec{a}\cdot\vec{b} - 2\vec{b}\cdot\vec{c} - 2\vec{c}\cdot\vec{a} = 9$

$\left(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}\right) = \frac{-3}{2}$

$\left|\vec{a} + \vec{b} + \vec{c}\right|^2 \geq 0$

$\left(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}\right) 3 + 2 \geq 0$

$\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} \geq \frac{-3}{2}$

Since चूंकि $= \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} = \frac{-3}{2}$

$\Rightarrow \left|\vec{a} + \vec{b} + \vec{c}\right| = 0 \Rightarrow \vec{a} + \vec{b} + \vec{c} = 0$

$\Rightarrow \left|2\vec{a} + 5(-\vec{a})\right| = |3\vec{a}| \Rightarrow 3$

15. The equation of a plane passing through the line of intersection of the planes $x + 2y + 3z = 2$ and $x - y + z = 3$ and at a distance $\frac{2}{\sqrt{3}}$ from the point (3, 1, –1) is **[IIT-JEE 2012, Paper-2, (3, –1), 66]**

एक समतल, जो समतलों $x + 2y + 3z = 2$ और $x - y + z = 3$ की प्रतिच्छेदी रेखा से गुजरता है और बिन्दु (3,1, –1) से $\frac{2}{\sqrt{3}}$ की दूरी पर है, का समीकरण निम्न है।

(A*) $5x - 11y + z = 17$ (B) $\sqrt{2}x + y = 3\sqrt{2} - 1$

(C) $x + y + z = \sqrt{3}$ (D) $x - \sqrt{2}y = 1 - \sqrt{2}$

Sol. **Ans. (A)**

Equation of required plane

$(x + 2y + 3z - 2) + \lambda(x - y + z - 3) = 0$

$\Rightarrow (1 + \lambda)x + (2 - \lambda)y + (3 + \lambda)z - (2 + 3\lambda) = 0$

distance from point (3, 1, – 1)

$$= \left| \frac{3+3\lambda+2-\lambda-3-\lambda-2-3\lambda}{\sqrt{(1+\lambda)^2+(2-\lambda)^2+(3+\lambda)^2}} \right| = \frac{2}{\sqrt{3}}$$

$\Rightarrow \quad \left| \frac{-2\lambda}{\sqrt{3\lambda^2+4\lambda+14}} \right| = \frac{2}{\sqrt{3}}$

$\Rightarrow \quad 3\lambda^2 = 3\lambda^2 + 4\lambda + 14$

$\Rightarrow \quad \lambda = -\frac{7}{2}$

equation of required plane

$5x - 11y + z - 17 = 0$

16. If $\vec{a}$ and $\vec{b}$ are vectors such that $|\vec{a}+\vec{b}| = \sqrt{29}$ and $\vec{a}\times(2\hat{i}+3\hat{j}+4\hat{k}) = (2\hat{i}+3\hat{j}+4\hat{k})\times\vec{b}$, then a possible value of $(\vec{a}+\vec{b})\cdot(-7\hat{i}+2\hat{j}+3\hat{k})$ is **[IIT-JEE 2012, Paper-2, (3, –1), 66]**

यदि सदिश $\vec{a}$ और $\vec{b}$ के लिए $|\vec{a}+\vec{b}| = \sqrt{29}$ और $\vec{a}\times(2\hat{i}+3\hat{j}+4\hat{k}) = (2\hat{i}+3\hat{j}+4\hat{k})\times\vec{b}$ है, तब $(\vec{a}+\vec{b})\cdot(-7\hat{i}+2\hat{j}+3\hat{k})$ का एक सम्भावित मान निम्न होगा–

(A) 0 (B) 3 (C*) 4 (D) 8

Hindi. Ans. (C)

Let माना $\vec{c} = 2\hat{i}+3\hat{j}+4\hat{k}$

$\vec{a}\times\vec{c} = \vec{c}\times\vec{b}$

$\Rightarrow \quad (\vec{a}+\vec{b})\times\vec{c} = \vec{0}$

$\Rightarrow \quad (\vec{a}+\vec{b}) \,||\, \vec{c}$

Let माना $(\vec{a}+\vec{b}) = \lambda\vec{c}$

$\Rightarrow \quad |\vec{a}+\vec{b}| = |\lambda|\,|\vec{c}|$

$\Rightarrow \quad \sqrt{29} = |\lambda| \,.\, \sqrt{29}$

$\Rightarrow \quad \lambda = \pm 1$

$\therefore \quad \vec{a}+\vec{b} = \pm(2\hat{i}+3\hat{j}+4\hat{k})$

Now अब $(\vec{a}+\vec{b})\cdot(-7\hat{i}+2\hat{j}+3\hat{k}) = \pm(-14+6+12)$

$= \pm 4$

17*. If the straight lines $\frac{x-1}{2} = \frac{y+1}{k} = \frac{z}{2}$ and $\frac{x+1}{5} = \frac{y+1}{2} = \frac{z}{k}$ are coplanar, then the plane(s) containing these two lines is(are) **[IIT-JEE 2012, Paper-2, (4, 0), 66]**

(A) $y + 2z = -1$ (B*) $y + z = -1$ (C*) $y - z = -1$ (D) $y - 2z = -1$

यदि सरल रेखायें $\frac{x-1}{2} = \frac{y+1}{k} = \frac{z}{2}$ और $\frac{x+1}{5} = \frac{y+1}{2} = \frac{z}{k}$ समतलीय (coplanar) हैं तो वह (वे) समतल जिसमें ये दोनों रेखायें स्थित हैं, निम्न है (हैं)

(A) $y + 2z = -1$ (B*) $y + z = -1$ (C*) $y - z = -1$ (D) $y - 2z = -1$

Sol. **Ans. (B,C)**

For co-planer lines $[\vec{a}-\vec{c}\quad \vec{b}\quad \vec{d}]=0$

$\vec{a} \equiv (1,-1,0)$, $= (-1,-1,0)$

$\vec{b} = 2\hat{i}+k\hat{j}+2\hat{k}$ $\qquad \vec{d} = 5\hat{i}+2\hat{j}+k\hat{k}$

Now $\begin{vmatrix} 2 & 0 & 0 \\ 2 & k & 2 \\ 5 & 2 & k \end{vmatrix} = 0 \quad \Rightarrow \quad k = \pm 2$

$\vec{n}_1 = \vec{b}_1 \times \vec{d}_1 = 6\hat{j}-6\hat{k}$ $\quad$ for $k = 2$

$\vec{n}_2 = \vec{b}_2 \times \vec{d}_2 = 14\hat{j}+14\hat{k}$ $\quad$ for $k = -2$

so the equation of planes are $(\vec{r}-\vec{a}).\vec{n}_1 = 0 \Rightarrow y - z = -1$ ...... (1)

$(\vec{r}-\vec{a}).\vec{n}_2 = 0 \Rightarrow y + z = -1$ ...... (2)

so answer is (B,C)

Hindi **Ans. (B,C)**

दो समतलीय रेखाओं के लिए $[\vec{a}-\vec{c}\quad \vec{b}\quad \vec{d}]=0$

$\vec{a} \equiv (1,-1,0)$, $= (-1,-1,0)$

$\vec{b} = 2\hat{i}+k\hat{j}+2\hat{k}$ $\qquad \vec{d} = 5\hat{i}+2\hat{j}+k\hat{k}$

अब $\begin{vmatrix} 2 & 0 & 0 \\ 2 & k & 2 \\ 5 & 2 & k \end{vmatrix} = 0 \quad \Rightarrow \quad k = \pm 2$

$\vec{n}_1 = \vec{b}_1 \times \vec{d}_1 = 6\hat{j}-6\hat{k}$ $\quad$ $k = 2$ के लिए

$\vec{n}_2 = \vec{b}_2 \times \vec{d}_2 = 14\hat{j}+14\hat{k}$ $\quad$ $k = -2$ के लिए

इसलिए समतल की समीकरण है $(\vec{r}-\vec{a}).\vec{n}_1 = 0 \Rightarrow y - z = -1$ ...... (1)

$(\vec{r}-\vec{a}).\vec{n}_2 = 0 \Rightarrow y + z = -1$ ...... (2)

इसलिए उत्तर (B,C) है।

18. Let $\overrightarrow{PR} = 3\hat{i}+\hat{j}-2\hat{k}$ and $\overrightarrow{SQ} = \hat{i}-3\hat{j}-4\hat{k}$ determine diagonals of a parallelogram PQRS and $\overrightarrow{PT} = \hat{i}+2\hat{j}+3\hat{k}$ be another vector. Then the volume of the parallelepiped determined by the vectors $\overrightarrow{PT}$, $\overrightarrow{PQ}$ and $\overrightarrow{PS}$ is **[JEE (Advanced) 2013, Paper-1, (2, 0)/60]**

मानाकि $\overrightarrow{PR} = 3\hat{i}+\hat{j}-2\hat{k}$ तथा $\overrightarrow{SQ} = \hat{i}-3\hat{j}-4\hat{k}$ एक समान्तर चतुर्भुज PQRS के विकर्ण निर्धारित करते है और $\overrightarrow{PT} = \hat{i}+2\hat{j}+3\hat{k}$ एक अन्य सदिश है, तब सदिशों $\overrightarrow{PT}$, $\overrightarrow{PQ}$ तथा $\overrightarrow{PS}$ द्वारा निर्धारित समान्तर षट्फलक का आयतन है– **[JEE (Advanced) 2013, Paper-1, (2, 0)/60]**

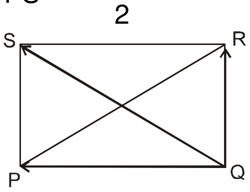
(A) 5 (B) 20 (C*) 10 (D) 30

Sol. **(C)**

$\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{PS}$

$\overrightarrow{SQ} = \overrightarrow{PQ} - \overrightarrow{PS}$

$\overrightarrow{PS} = \dfrac{\overrightarrow{PR} - \overrightarrow{SQ}}{2}$



$V = \left| \left[\overrightarrow{PQ} \quad \overrightarrow{PS} \quad \overrightarrow{PT} \right] \right|$

$V = \frac{1}{4} \left| \left[\overrightarrow{PR} + \overrightarrow{SQ} \,,\, \overrightarrow{PR} - \overrightarrow{SQ} \,,\, \overrightarrow{PT} \right] \right|$

$V = \frac{1}{2} \left| \left[\overrightarrow{PR} \,,\, \overrightarrow{SQ} \,,\, \overrightarrow{PT} \right] \right|$

$\frac{1}{2} \begin{vmatrix} 3 & 1 & -2 \\ 1 & -3 & -4 \\ 1 & 2 & 3 \end{vmatrix}$

$\frac{1}{2}(-3-7-10) = 10$

19. Perpendicular are drawn from points on the line $\frac{x+2}{2} = \frac{y+1}{-1} = \frac{z}{3}$ to the plane x + y + z = 3. The feet of perpendiculars lie on the line **[JEE (Advanced) 2013, Paper-1, (2, 0)/60]**

तल x + y + z = 3 पर रेखा $\frac{x+2}{2} = \frac{y+1}{-1} = \frac{z}{3}$ पर स्थित बिन्दुओं से लम्ब डाले जाते है। लम्ब-पाद निम्न रेखा पर स्थित है–

(A) $\frac{x}{5} = \frac{y-1}{8} = \frac{z-2}{-13}$ (B) $\frac{x}{2} = \frac{y-1}{3} = \frac{z-2}{-5}$ (C) $\frac{x}{4} = \frac{y-1}{3} = \frac{z-2}{-7}$ (D*) $\frac{x}{2} = \frac{y-1}{-7} = \frac{z-2}{5}$

Sol. **(D)**

Any point on line $\frac{x+2}{2} = \frac{y+1}{-1} = \frac{z}{3} = \lambda$

Let any two points on this line are

A(– 2, – 1, 0), B (0, – 2, 3) Put (λ = 0, 1)

Let foot of perpendicular from A(– 2, – 1, 0) on plane is (α, β, γ)

$\Rightarrow \quad \frac{\alpha+2}{1} = \frac{\beta+1}{1} = \frac{\beta-0}{1} = \mu$ (say)

Also, $\alpha + \beta + \gamma = 3$

$\Rightarrow \quad \mu - 2 + \mu - 1 + \mu = 3 \Rightarrow \mu = 2$

$\Rightarrow$ M(0, 1, 2)

Similarly foot of perpendicular from B(0, – 2, 3) on plane is N $\left(\frac{2}{3}, \frac{-4}{3}, \frac{11}{3}\right)$

So, equation of MN is $\frac{x-0}{\frac{2}{3}} = \frac{y-1}{\frac{-7}{3}} = \frac{z-2}{\frac{5}{3}}$.

Hindi. **(D)**

रेखा $\frac{x+2}{2} = \frac{y+1}{-1} = \frac{z}{3} = \lambda$ पर कोई बिन्दु

माना इस रेखा पर कोई दो बिन्दु है

A(– 2, – 1, 0), B (0, – 2, 3) (λ = 0, 1) रखने पर

माना A(– 2, – 1, 0) से समतल पर लम्बपाद के निर्देशांक (α, β, γ) है।

$\Rightarrow \quad \frac{\alpha+2}{1} = \frac{\beta+1}{1} = \frac{\beta-0}{1} = \mu$ (माना)

तथा, $\alpha + \beta + \gamma = 3$

$\Rightarrow \quad \mu - 2 + \mu - 1 + \mu = 3 \Rightarrow \mu = 2$

$\Rightarrow$ M(0, 1, 2)

इसी प्रकार B(0, – 2, 3) से समतल पर लम्बपाद के निर्देशांक N $\left(\frac{2}{3}, \frac{-4}{3}, \frac{11}{3}\right)$ है।

इसलिए MN का समीकरण $\frac{x-0}{\frac{2}{3}} = \frac{y-1}{\frac{-7}{3}} = \frac{z-2}{\frac{5}{3}}$.

20.* A line l passing through the origin is perpendicular to the lines

$l_1 : (3 + t)\ \hat{i} + (-1 + 2t)\ \hat{j} + (4 + 2t)\ \hat{k}, -\infty < t < \infty$

$l_2 : (3 + 2s)\ \hat{i} + (3 + 2s)\ \hat{j} + (2 + s)\ \hat{k}, -\infty < s < \infty$

Then, the coordinate(s) of the point(s) on l_2 at a distance of $\sqrt{17}$ from the point of intersection of l and l_1 is(are) **[JEE (Advanced) 2013, Paper-1, (4, – 1)/60]**

(A) $\left(\frac{7}{3}, \frac{7}{3}, \frac{5}{3}\right)$ (B*) (–1, ,–1, 0) (C) (1, 1, 1) (D*) $\left(\frac{7}{9}, \frac{7}{9}, \frac{8}{9}\right)$

एक रेखा l, जो मूलबिन्दु से गुजरती है, रेखाओं

$l_1 : (3 + t)\ \hat{i} + (-1 + 2t)\ \hat{j} + (4 + 2t)\ \hat{k}, -\infty < t < \infty$

$l_2 : (3 + 2s)\ \hat{i} + (3 + 2s)\ \hat{j} + (2 + s)\ \hat{k}, -\infty < s < \infty$

पर लम्बवत है। तब, l_2 पर स्थित बिन्दु (बिन्दुओं) के निर्देशांक, जो रेखाओं l तथा l_1 के प्रतिच्छेद बिन्दु से $\sqrt{17}$ की दूरी पर हैं (हैं), निम्न है (हैं) : **[JEE (Advanced) 2013, Paper-1, (4, – 1)/60]**

(A) $\left(\frac{7}{3}, \frac{7}{3}, \frac{5}{3}\right)$ (B*) (–1, ,–1, 0) (C) (1, 1, 1) (D*) $\left(\frac{7}{9}, \frac{7}{9}, \frac{8}{9}\right)$

Sol. **(B,D)**

Let equation of line ℓ is

$$\ell : \frac{x-0}{a} = \frac{y-0}{b} = \frac{z-0}{c} = k$$

This line ℓ is perpendicular to given line ℓ_1 and ℓ_2.

Hence $a + 2b + 2c = 0$

$2a + 2b + c = 0$

$$\frac{a}{-2} = \frac{b}{3} = \frac{c}{-2}$$

Hence equation of ℓ is $\frac{x}{-2} = \frac{y}{3} = \frac{z}{-2} = k_1,\ k_2$

$\downarrow \quad \downarrow$

for ℓ_1 for ℓ_2

Now $A(-2k_1, 3k_1, -2k_1)$ $B(-2k_2, 3k_2, -2k_2)$

Point A satisfied ℓ_1

$$-2k_1\ \hat{i} + 3k_1\hat{j} - 2k_1\hat{k} = (3 + t)\ \hat{i} + (-1 + 2t)\ \hat{j} + (4 + 2t)\ \hat{k}$$

$3 + t = -2k_1$ ....(1)

$-1 + 2t = 3k_1$ ....(2)

$4 + 2t = -2k_1$ ....(3)

(2) & (3) $-5 = 5k_1 \Rightarrow k_1 = -1 \Rightarrow A\ (2, -3, 2)$

Let any point on ℓ_2 (3 + 2S, 3 + 2S, 2 + S)

Given $\sqrt{(1+2S)^2 + (6+2S)^2 + (S)^2} = \sqrt{17}$

$9S^2 + 28S + 37 = 17$

$9S^2 + 28S + 20 = 0$

$9S^2 + 18S + 10S + 20 = 0$

$9S(S + 2) + 10 (S + 2) = 0$
$S = -2, -10/9$
Hence $(-1, -1, 0)$, $(7/9, 7/9, 8/9)$
Ans. (B) & (D)

Hindi **(B,D)**

माना रेखा ℓ का समीकरण

$$\ell : \frac{x-0}{a} = \frac{y-0}{b} = \frac{z-0}{c} = k$$

यह रेखा ℓ दी गई रेखा ℓ_1 तथा ℓ_2 के लम्बवत् है

अतः $a + 2b + 2c = 0$
$2a + 2b + c = 0$
$$\frac{a}{-2} = \frac{b}{3} = \frac{c}{-2}$$

अतः ℓ का समीकरण $\frac{x}{-2} = \frac{y}{3} = \frac{z}{-2} = \underset{\text{for } \ell_1}{k_1}, \quad \underset{\text{for } \ell_2}{k_2}$

अब $A(-2k_1, 3k_1, -2k_1)$ $B(-2k_2, 3k_2, -2k_2)$

बिन्दु A, ℓ_1 पर संतुष्ट होता है

$$-2k_1\hat{i} + 3k_1\hat{j} - 2k_1\hat{k} = (3 + t)\hat{i} + (-1 + 2t)\hat{j} + (4 + 2t)\hat{k}$$

$3 + t = -2k_1$ ....(1)
$-1 + 2t = 3k_1$ ....(2)
$4 + 2t = -2k_1$ ....(3)

(2) & (3) $-5 = 5k_1 \Rightarrow k_1 = -1 \Rightarrow A (2, -3, 2)$

माना ℓ_2 $(3 + 2S, 3 + 2S, 2 + S)$ पर कोई बिन्दु

दिया गया है $\sqrt{(1+2S)^2 + (6+2S)^2 + (S)^2} = \sqrt{17}$

$9S^2 + 28S + 37 = 17$
$9S^2 + 28S + 20 = 0$
$9S^2 + 18S + 10S + 20 = 0$
$9S(S + 2) + 10 (S + 2) = 0$
$S = -2, -10/9$

अतः $(-1, -1, 0)$, $(7/9, 7/9, 8/9)$
Ans. (B) & (D)

21. Consider the set of eight vectors $V = \{a\hat{i} + b\hat{j} + c\hat{k} \ : \ a,b,c \in \{-1,1\}\}$. Three non-coplanar vectors can be chosen from V in 2^p ways. Then p is **[JEE (Advanced) 2013, Paper-1, (4, – 1)/60]**

आठ सदिशों का समुच्चय $V = \{a\hat{i} + b\hat{j} + c\hat{k} \ : \ a,b,c \in \{-1,1\}\}$. लीजिए । V से तीन असमतलीय सदिश 2^p प्रकार से चुने जा सकते है। तब p का मान है: **[JEE (Advanced) 2013, Paper-1, (4, – 1)/60]**

Ans. ${}^8C_3 - 24 = 32$

Sol. Among set of eight vectors four vectors form body diagonals of a cube, remaining four will be parallel (unlike) vectors.
Numbers of ways of selecting three vectors will
be ${}^4C_3 \times 2 \times 2 \times 2 = 2^5$
Hence p = 5

Alternative

Eight vectors
$\vec{x} \equiv \hat{i} + \hat{j} + \hat{k}$

$\vec{y} \equiv \hat{i} + \hat{j} - \hat{k}$

$\vec{z} \equiv \hat{i} - \hat{j} + \hat{k}$

$\vec{\omega} = \hat{i} - \hat{j} - \hat{k}$

$\vec{x}' = -\hat{i} - \hat{j} - \hat{k}$

$\vec{y}' = -\hat{i} - \hat{j} + \hat{k}$

$\vec{z}' = -\hat{i} + \hat{j} - \hat{k}$

$\vec{\omega}' = -\hat{i} + \hat{j} + \hat{k}$

If we take $\vec{x}$, $\vec{x}'$ and any one of remaining sin x, vectors will always be coplaner

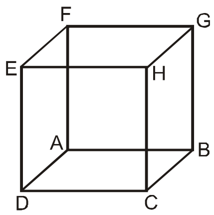
$\therefore$ No. of coplaner vectors = 6

similarly on taking $\vec{y}$, $\vec{y}'$ = 6

z, $\vec{z}'$ = 6

ω, $\vec{\omega}'$ = 6 $\therefore$ No. of set of coplaner vectors = 24

Alternative



A(0, 0, 0)
B(1, 0, 0)
C(1, 0, 1)
D(0, 0, 1)
E(0, 1, 1)
F(0, 1, 0)
G(1, 1, 0)
H(1, 1, 1)

$\overrightarrow{AH} = \hat{i} + \hat{j} + \hat{k}$

$\overrightarrow{BE} = -\hat{i} + \hat{j} + \hat{k}$

$\overrightarrow{CF} = -\hat{i} + \hat{j} - \hat{k}$

Non-coplaner

Hindi. आठ सदिशों के समुच्चयों से चार सदिश, घन के विकर्ण के शीर्ष है। शेष, सभी समान्तर सदिश है। तीन सदिशों के चयन करने के क्रमचय होगें–

$^4C_3 \times 2 \times 2 \times 2 = 2^5$

अतः p = 5

Alternative

आठ सदिश

$\vec{x} \equiv \hat{i} + \hat{j} + \hat{k}$

$\vec{y} \equiv \hat{i} + \hat{j} - \hat{k}$

$\vec{z} \equiv \hat{i} - \hat{j} + \hat{k}$

$\vec{\omega} = \hat{i} - \hat{j} - \hat{k}$

$\vec{x}' = -\hat{i} - \hat{j} - \hat{k}$

$\vec{y}' = -\hat{i} - \hat{j} + \hat{k}$

$\vec{z}' = -\hat{i} + \hat{j} - \hat{k}$

$\vec{\omega}' = -\hat{i} + \hat{j} + \hat{k}$

यदि हम $\vec{x}$, $\vec{x}'$ तथा sin x मे से एक शेष सदिश, सदैव समतलीय होगें।

$\therefore$ समतल सदिशों की संख्या = 6

इसी प्रकार $\vec{y}$, $\vec{y}'$ = 6 लेने पर

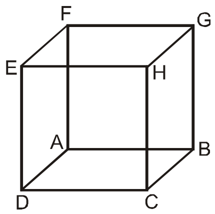
z, $\vec{z}'$ = 6

ω, $\vec{\omega}'$ = 6 $\therefore$ समतलीय सदिशों के समुच्चयों की संख्या = 24

Ans. $^8C_3 - 24 = 32$

Alternative

A(0, 0, 0)
B(1, 0, 0)



C(1, 0, 1)
D(0, 0, 1)
E(0, 1, 1)
F(0, 1, 0)
G(1, 1, 0)
H(1, 1, 1)

$\overrightarrow{AH} = \hat{i} + \hat{j} + \hat{k}$

$\overrightarrow{BE} = -\hat{i} + \hat{j} + \hat{k}$

$\overrightarrow{CF} = -\hat{i} + \hat{j} - \hat{k}$ असमतलीय

22.* Two lines $L_1 : x = 5, \frac{y}{3-\alpha} = \frac{z}{-2}$ and $L_2 : x = \alpha, \frac{y}{-1} = \frac{z}{2-\alpha}$ are coplanar. Then α can take value(s)

दो रेखाएं $L_1 : x = 5, \frac{y}{3-\alpha} = \frac{z}{-2}$ तथा $L_2 : x = \alpha, \frac{y}{-1} = \frac{z}{2-\alpha}$ समतलीय है। तब α का मान हो सकता है–

[JEE (Advanced) 2013, Paper-2, (3, –1)/60]

(A*) 1 (B) 2 (C) 3 (D*) 4

Sol. **(A, D)**

$$\frac{x-5}{0} = \frac{-y}{\alpha-3} = \frac{z}{-2}$$

$$\frac{x-\alpha}{0} = \frac{y}{-1} = \frac{z}{2-\alpha}$$

$$\begin{vmatrix} 5-\alpha & 0 & 0 \\ 0 & 3-\alpha & -2 \\ 0 & -1 & 2-\alpha \end{vmatrix} = 0$$

$(5 - \alpha)\ ((3 - \alpha)\ (2 - \alpha) - 2) = 0$

$(\alpha^2 - 5\alpha + 6 - 2) = 0$

$(\alpha - 5)(\alpha^2 - 5\alpha + 4) = 0$

$\alpha = 1, 4, 5$

23. Match List I with List II and select the correct answer using the code given below the lists :

[JEE (Advanced) 2013, Paper-2, (3, –1)/60]

	List - I	**List - II**
P.	Volume of parallelopiped determined by vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is 2. Then the volume of the parallelepiped determined by vectors $2(\vec{a}\times\vec{b}), 3(\vec{b}\times\vec{c})$ and $(\vec{c}\times\vec{a})$ is	1. 100
Q.	Volume of parallelepiped determined by vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is 5. Then the volume of the parallelepiped determined by vectors $3(\vec{a}+\vec{b}), (\vec{b}+\vec{c})$ and $2(\vec{c}+\vec{a})$ is	2. 30
R.	Area of a triangle with adjacent sides determined by vectors $\vec{a}$ and $\vec{b}$ is 20. Then the area of the triangle with adjacent sides determined by vectors $\left(2\vec{a}+3\vec{b}\right)$ and $(\vec{a}-\vec{b})$ is	3. 24
S.	Area of a paralelogram with adjacent sides determined by vectors $\vec{a}$ and $\vec{b}$ is 30. Then the area of the parallelogram with adjacent sides determined by vectors $(\vec{a}+\vec{b})$ and $\vec{a}$ is	4. 60

सूची I को सूची II से सुमेलित कीजिए तथा सूचियों के नीचे दिए गए कोड का प्रयोग करके सही उत्तर चुनिये :

[JEE (Advanced) 2013, Paper-2, (3, –1)/60]

	सूची- I	**सूची - II**
P.	सदिशों $\vec{a}, \vec{b}$ तथा $\vec{c}$ द्वारा निर्धारित समांतर षट्फलक का आयतन 2 है। तब सदिशों $2(\vec{a}\times\vec{b}), 3(\vec{b}\times\vec{c})$ तथा $(\vec{c}\times\vec{a})$ द्वारा निर्धारित समांतर षट्फलक का आयतन है	1. 100
Q.	सदिशों $\vec{a}, \vec{b}$ तथा $\vec{c}$ द्वारा निर्धारित समांतर षट्फलक का आयतन 5 है। तब सदिशों $3(\vec{a}+\vec{b}), (\vec{b}+\vec{c})$ तथा 2 $(\vec{c}+\vec{a})$ द्वारा निर्धारित समांतर षट्फलक का आयतन है	2. 30
R.	एक त्रिभुज का क्षेत्रफल, जिसकी संलग्न भुजाएँ सदिशों $\vec{a}$ तथा $\vec{b}$ द्वारा निर्धारित है, 20 है। तब सदिशों $\left(2\vec{a}+3\vec{b}\right)$ तथा $(\vec{a}-\vec{b})$ द्वारा निर्धारित संलग्न भुजाओं वाले त्रिभुज का क्षेत्रफल है	3. 24
S.	एक समांतर चतुर्भुज का क्षेत्रफल, जिसकी संलग्न भुजाएँ सदिशों $\vec{a}$ तथा $\vec{b}$ द्वारा निर्धारित हैं 30 है। तब सदिशों $(\vec{a}+\vec{b})$ तथा $\vec{a}$ द्वारा निर्धारित संलग्न भुजाओं वाले समांतर चतुर्भुज का क्षेत्रफल है	4. 60

Codes :

	P	Q	R	S
(A)	4	2	3	1
(B)	2	3	1	4
(C*)	3	4	1	2
(D)	1	4	3	2

Sol. **(C)**

(P) $[\vec{a}\ \ \vec{b}\ \ \vec{c}] = 2$

$2\left(\vec{a} \times \vec{b}\right), 3\left(\vec{b}\times\vec{c}\right), \left(\vec{c}\times\vec{a}\right)$

$6\,[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}\ ] = 6\ [\vec{a}\,\vec{b}\,\vec{c}]^2$

$= 6 \times 4 = 24$

$P \to 3$

(Q) $[\vec{a}\ \ \vec{b}\ \ \vec{c}] = 5$

$[3\ (\vec{a}+\vec{b})\ \ (\vec{b}+\vec{c})\ \ 2(\vec{c}+\vec{a})\,]$

$= 6 \times 2[\vec{a}\ \ \vec{b}\ \ \vec{c}]$

$= 12 \times 5 = 60$

$Q \to 4$

(R) $\frac{1}{2}\,|\vec{a}\times\vec{b}| = 20$

$\Delta_1 = \frac{1}{2}\ \ |(2\vec{a}+3\vec{b}) \times (\vec{a}-\vec{b})\ |$

$= \ |-2\vec{a}\times\vec{b}-3(\vec{a}\times\vec{b})\ |$

$= \frac{5}{2}\,|\vec{a}\times\vec{b}|$

$= 5 \times 20 = 100$

$R \to 1$

(S) $|\vec{a}\times\vec{b}| = 30$

$|(\vec{a}+\vec{b}) \times \ \vec{a}\,| = |\vec{b}\times\vec{a}| = 30$

$S \to 2$

24. Consider the lines $L_1 : \frac{x-1}{2} = \frac{y}{-1} = \frac{z+3}{1}$, $L_2\ \frac{x-4}{1} = \frac{y+3}{1} = \frac{z+3}{2}$: and the planes $P_1 : 7x + y + 2z = 3$, $P_2 : 3x + 5y - 6z = 4$. Let $ax + by + cz = d$ the equation of the plane passing through the point of intersection of lines L_1 and L_2, and perpendicular to planes P_1 and P_2.
Match List - I with List- II and select the correct answer using the code given below the lists :
[JEE (Advanced) 2013, Paper-2, (3, –1)/60]

रेखाएं $L_1 : \frac{x-1}{2} = \frac{y}{-1} = \frac{z+3}{1}$, $L_2\ \frac{x-4}{1} = \frac{y+3}{1} = \frac{z+3}{2}$: तथा समतल $P_1 : 7x + y + 2z = 3$, $P_2 : 3x + 5y - 6z = 4$ लीजिए। माना कि $ax + by + cz = d$, रेखाओं L_1 व L_2 के प्रतिच्छेद बिन्दु से गुजरने वाला तथा समतल P_1 व P_2 के लम्बवत्, समतल का समीकरण है।
सूची - I को सूची - II से सुमेलित कीजिए तथा सूचियों के नीचे दिए गए कोड का प्रयोग करके सही उत्तर चुनिये :
[JEE (Advanced) 2013, Paper-2, (3, –1)/60]

List- I / सूची I		**List- II / सूची II**	
P.	a =	1.	13
Q.	b =	2.	–3
R.	c =	3.	1
S.	d =	4.	–2

Codes :

	P	Q	R	S
(A*)	3	2	4	1
(B)	1	3	4	2
(C)	3	2	1	4

(D) 2 4 1 3

Sol. **(A)**

$$L_1 : \frac{x-1}{2} = \frac{y}{-1} = \frac{z+3}{1}$$

Normal of plane P : $\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & 1 & 2 \\ 3 & 5 & -6 \end{vmatrix}$

$$= \hat{i}\ (-16) - \hat{j}\ (-42-6) + \hat{k}\ (32)$$

$$= -16\ \hat{i} + 48\hat{j} + 32\hat{k}$$

$$\Rightarrow \vec{n} = \hat{i} - 3\hat{j} - 2\hat{k}$$

Point of intersection of L_1 and L_2

$2k_1 + 1 = k_2 + 4$

$-k_1 = k_2 - 3$

$1 = 3k_2 - 2$

$k_2 = 1$

Point of intersection (5, –2, –1)

Plane $(x-5) - 3(y+7) - 2(z+1) = 0$

$x - 3y - 2z - 5 - 6 - 2 = 0$

$x - 3y - 2z = 13$

$\Rightarrow$ $a = 1, b = 3, c = -2, d = 13$

Hindi **(A)**

$$L_1 : \frac{x-1}{2} = \frac{y}{-1} = \frac{z+3}{1}$$

समतल P का अभिलम्ब : $\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & 1 & 2 \\ 3 & 5 & -6 \end{vmatrix}$

$$= \hat{i}\ (-16) - \hat{j}\ (-42-6) + \hat{k}\ (32)$$

$$= -16\ \hat{i} + 48\hat{j} + 32\hat{k}$$

$$\Rightarrow \vec{n} = \hat{i} - 3\hat{j} - 2\hat{k}$$

L_1 तथा L_2 का प्रतिच्छेद बिन्दु

$2k_1 + 1 = k_2 + 4$

$-k_1 = k_2 - 3$

$1 = 3k_2 - 2$

$k_2 = 1$

प्रतिच्छेद बिन्दु (5, –2, –1)

समतल $(x-5) - 3(y+7) - 2(z+1) = 0$

$x - 3y - 2z - 5 - 6 - 2 = 0$

$x - 3y - 2z = 13$

$\Rightarrow$ $a = 1, b = 3, c = -2, d = 13$

25*. Let $\vec{x}, \vec{y}$ and $\vec{z}$ be three vectors each of magnitude $\sqrt{2}$ and the angle between each pair of them is $\frac{\pi}{3}$. If $\vec{a}$ is a nonzero vector perpendicular to $\vec{x}$ and $\vec{y} \times \vec{z}$ and $\vec{b}$ is a nonzero vector perpendicular to $\vec{y}$ and $\vec{z} \times \vec{x}$, then **[JEE (Advanced) 2014, Paper-1, (3, 0)/60]**

(A*) $\vec{b} = (\vec{b}.\vec{z})(\vec{z} - \vec{x})$ (B*) $\vec{a} = (\vec{a}.\vec{y})(\vec{y} - \vec{z})$ (C*) $\vec{a}.\vec{b} = -(\vec{a}.\vec{y})(\vec{b}.\vec{z})$ (D) $\vec{a} = (\vec{a}.\vec{y})(\vec{z} - \vec{y})$

माना कि सदिशों (vectors) $\vec{x}, \vec{y}$ तथा $\vec{z}$ में प्रत्येक का परिमाण $\sqrt{2}$ हैं तथा प्रत्येक युग्म (pair) के मध्य का कोण $\frac{\pi}{3}$ है। यदि शून्येतर (non-zero) सदिश $\vec{a}$ सदिशों $\vec{x}$ तथा $\vec{y}\times\vec{z}$ के लम्बवत् (perpendicular) है एवं शून्येतर सदिश $\vec{b}$ सदिशों $\vec{y}$ तथा $\vec{z}\times\vec{x}$ के लम्बवत् है, तब

(A*) $\vec{b} = (\vec{b}.\vec{z})(\vec{z}-\vec{x})$ (B*) $\vec{a} = (\vec{a}.\vec{y})(\vec{y}-\vec{z})$ (C*) $\vec{a}.\vec{b} = -(\vec{a}.\vec{y})(\vec{b}.\vec{z})$ (D) $\vec{a} = (\vec{a}.\vec{y})(\vec{z}-\vec{y})$

Ans. **(ABC)**

Sol. $|\vec{x}| = |\vec{y}| = |\vec{z}| = \sqrt{2}$

$\theta = \frac{\pi}{3}$

$\vec{a} = \lambda\vec{x}\times(\vec{y}\times\vec{z})$

$\vec{b} = \mu\vec{y}\times(\vec{z}\times\vec{x})$

$\vec{a} = ((\vec{x}.\vec{z})\vec{y} - (\vec{x}.\vec{y})\vec{z})$

$\vec{a} = \lambda\left(2\times\frac{1}{2}\vec{y} - 2\times\frac{1}{2}\vec{z}\right)$

$\vec{a} = \lambda(\vec{y}-\vec{z})$

$\vec{b} = \mu(\vec{z}-\vec{x})$

Similarly इसी प्रकार

$\vec{a}.\vec{y} = \lambda\left(2 - 2\times\frac{1}{2}\right) = \lambda$

$\vec{a} = (\vec{a}.\vec{y})(\vec{y}-\vec{z}) \quad \Rightarrow \quad$ (B)

$\vec{b}.\vec{z} = \mu\left(2 - 2\times\frac{1}{2}\right)$

$\mu = \vec{b}.\vec{z}$

$\therefore\ \vec{b} = (\vec{b}.\vec{z})(\vec{z}-\vec{x}) \quad \Rightarrow \quad$ (A)

(A) $\vec{a}.\vec{b} = (\vec{a}.\vec{y})(\vec{y}-\vec{z}).(\vec{b}.\vec{y})(\vec{z}-\vec{x})$

$= (\vec{a}.\vec{y})(\vec{b}.\vec{z})(\vec{y}\vec{z} - \vec{y}\vec{x} - 2 + \vec{x}\vec{z})$

$= (\vec{a}.\vec{y})(\vec{b}.\vec{z})$

$= -\ (\vec{a}.\vec{y})(\vec{b}.\vec{z}) \quad \Rightarrow \quad$ (C)

26. From a point $P(\lambda,\lambda,\lambda)$, perpendiculars PQ and PR are drawn respectively on the lines $y = x, z = 1$ and $y = -x, z = -1$. If P is such that $\angle QPR$ is a right angle, then the possible value(s) of λ is(are)

[JEE (Advanced) 2014, Paper-1, (3, 0)/60]

(A) $\sqrt{2}$ (B) 1 (C*) -1 (D) $-\sqrt{2}$

बिन्दु $P(\lambda,\lambda,\lambda)$ से रेखाओं $y = x, z = 1$ तथा $y = -x, z = -1$ पर डाले गये लम्ब (perpendicular) क्रमशः PQ तथा PR हैं। यदि $\angle QPR$ समकोण (right angle) है, तो λ का(के) संभावित मान है(हैं)

(A) $\sqrt{2}$ (B) 1 (C*) -1 (D) $-\sqrt{2}$

Ans. **(C)**

Sol. Line is

$\frac{x-0}{1} = \frac{y-0}{1} = \frac{z-0}{1} = \alpha \qquad ......(1)$

$Q(\alpha, \alpha, 1)$

Direction ratio of PQ are

$\lambda - \alpha, \lambda - \alpha, \lambda - 1$

Since PQ is perpendicular to (1)

$\therefore$ $\lambda - \alpha + \lambda - \alpha + 0 = 0$

$\lambda = \alpha$

$\therefore$ Direction ratio of PQ are

$0, 0, \lambda - 1$

Another line is

$$\frac{x-0}{-1} = \frac{y-0}{1} = \frac{z+1}{0} = \beta \qquad ......(2)$$

$\therefore$ R $(-\beta, \beta, -1)$

$\therefore$ Direction ratio of PR are

$\lambda + \beta, \lambda - \beta, \lambda + 1$

Since PQ is perpendicular to (ii)

$\therefore$ $-\lambda - \beta + \lambda - \beta = 0$

$\beta = 0$

$\therefore$ R(0, 0, – 1)

and Direction ratio of PQ are $\lambda, \lambda, \lambda + 1$

Since $PQ \perp PR$

$\therefore$ $0 + 0 + \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1 \Rightarrow$ B, C

For $\lambda = 1$ the point is on the line so it will be rejected.

$\Rightarrow$ $\lambda = -1$.

Hindi. रेखा

$$\frac{x-0}{1} = \frac{y-0}{1} = \frac{z-0}{1} = \alpha \qquad ......(1)$$

Q(α, α, 1)

PQ के दिक अनुपात

$\lambda - \alpha, \lambda - \alpha, \lambda - 1$

चूंकि PQ, (1) के लम्बवत् है

$\therefore$ $\lambda - \alpha + \lambda - \alpha + 0 = 0$

$\lambda = \alpha$

$\therefore$ PQ के दिक अनुपात

$0, 0, \lambda - 1$

अन्य रेखा है

$$\frac{x-0}{-1} = \frac{y-0}{1} = \frac{z+1}{0} = \beta \qquad ......(2)$$

$\therefore$ R $(-\beta, \beta, -1)$

$\therefore$ PR के दिक अनुपात

$\lambda + \beta, \lambda - \beta, \lambda + 1$

चूंकि PQ, (ii) के लम्बवत् है

$\therefore$ $-\lambda - \beta + \lambda - \beta = 0$

$\beta = 0$

$\therefore$ R(0, 0, – 1) और PQ के दिक् अनुपात $\lambda, \lambda, \lambda + 1$ है

चूंकि $PQ \perp PR$ $\therefore$ $0 + 0 + \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1 \Rightarrow$ B, C

$\lambda = 1$ के लिए बिन्दु रेखा पर है इसलिए यह अस्वीकार्य होगा।

$\Rightarrow$ $\lambda = -1$.

27. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three non-coplanar unit vectors such that the angle between every pair of them is $\frac{\pi}{3}$. If $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}=p\vec{a}+q\vec{b}+r\vec{c}$, where p,q and r are scalars, then the value of $\frac{p^2+2q^2+r^2}{q^2}$ is

माना कि $\vec{a}$, $\vec{b}$ तथा $\vec{c}$ तीन असमतलीय (non-coplanar) इकाई सदिश है, जिनके प्रत्येक युग्म के मध्य का कोण $\frac{\pi}{3}$ है। यदि $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}=p\vec{a}+q\vec{b}+r\vec{c}$ जहाँ p,q एवम् r अदिश (scalars) है, तब $\frac{p^2+2q^2+r^2}{q^2}$ का मान है :

[JEE (Advanced) 2014, Paper-1, (3, 0)/60]

Ans. **(4)**

Sol. $p\vec{a} + q\vec{b} + r\vec{c} = a \times b + b \times c$

Taking dot product with $\vec{a}, \vec{b}, \vec{c}$ we get ($\vec{a}, \vec{b}, \vec{c}$ के साथ अदिश गुणन लेने पर)

$$p + \frac{q}{2} + \frac{r}{2} = [a\ b\ c] \quad ......(1)$$

$$\frac{p}{2} + q + \frac{r}{2} = 0 \quad ......(2)$$

$$\frac{p}{2} + \frac{q}{2} + r = [a\ b\ c] \quad ......(3)$$

(1) & (3) $\Rightarrow$ p = r & q = –p (1) और (3) $\Rightarrow$ p = r & q = –p

$$\frac{p^2+2q^2+r^2}{q^2} = \frac{p^2+2p^2+p^2}{p^2} = 4 \text{ Ans.}$$

28. **List I** **List II**

[JEE (Advanced) 2014, Paper-2, (3, –1)/60]

P. Let $y(x) = \cos(3\cos^{-1} x)$, $x \in [-1, 1]$, $x \neq \pm\frac{\sqrt{3}}{2}$. Then $\frac{1}{y(x)}\left\{(x^2-1)\frac{d^2y(x)}{dx^2}+x\frac{dy(x)}{dx}\right\}$ equals — **1.** 1

Q. Let $A_1, A_2, ..., A_n$ (n > 2) be the vertices of a regular polygon of n sides with its centre at the origin. Let $\vec{a}_k$ be the position vector of the point A_k, k = 1, 2,...., n. If $\left|\sum_{k=1}^{n-1}(\vec{a}_k \times \vec{a}_{k+1})\right| = \left|\sum_{k=1}^{n-1}(\vec{a}_k . \vec{a}_{k+1})\right|$, then the minimum value of n is — **2.** 2

R. If the normal from the point P(h, 1) on the ellipse $\frac{x^2}{6}+\frac{y^2}{3}=1$ is perpendicular to the line x + y = 8, then the value of h is — **3.** 8

S. Number of positive solutions satisfying the equation is $\tan^{-1}\left(\frac{1}{2x+1}\right)+\tan^{-1}\left(\frac{1}{4x+1}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$ — **4.** 9

सूची- I　　　　　　　　　　　　　　　　　　　　　　　　　　सूची - II

[JEE (Advanced) 2014, Paper-2, (3, –1)/60]

P. मानाकि $y(x) = \cos(3\cos^{-1}x)$, $x \in [-1, 1]$, $x \neq \pm\frac{\sqrt{3}}{2}$, तो $\frac{1}{y(x)}\left\{(x^2-1)\frac{d^2y(x)}{dx^2}+x\frac{dy(x)}{dx}\right\}$ का मान है– 　　**1.** 1

Q. मानाकि $A_1, A_2, \ldots\ldots, A_n$ $(n > 2)$ एक n भुजीय समबहुभुज (regular polygon) के शीर्ष (vertices) है जिसका केन्द्र मूलबिन्दु में है। माना कि $\vec{a}_k$ बिन्दु A_k, $k = 1, 2, \ldots, n$ का स्थिति सदिश (position vector) है। यदि $\left|\sum_{k=1}^{n-1}(\vec{a}_k \times \vec{a}_{k+1})\right| = \left|\sum_{k=1}^{n-1}(\vec{a}_k . \vec{a}_{k+1})\right|$ है, तब n का न्यूनतम मान है– 　　**2.** 2

R. यदि दीर्घवृत्त (ellipse) $\frac{x^2}{6}+\frac{y^2}{3}=1$ पर बिन्दु P(h, 1) से खींचा गया अभिलम्ब रेखा x + y = 8 पर लम्बवत् है, तो h का मान है– 　　**3.** 8

S. समीकरण $\tan^{-1}\left(\frac{1}{2x+1}\right)+\tan^{-1}\left(\frac{1}{4x+1}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$ को संतुष्ट करने वाले धनात्मक हलों की संख्या है– 　　**4.** 9

	P	Q	R	S
(A)	4	3	2	1
(B)	2	4	3	1
(C)	4	3	1	2
(D)	2	4	1	3

Ans. (A)

Sol. (P) $y = 4x^3 - 3x$ 　　where $\cos\theta = x$

$\frac{dy}{dx} = 12x^2 - 3$

$\frac{d^2y}{dx^2} + x\frac{dy}{dx} = (x^2 - 1) . 24x + x(12x^2 - 3)$

$= 36x^3 - 27x = 9(4x^3 - 3x) = 9y$

Hence $\frac{1}{y}\left\{(x^2-1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}\right\} = 9$

(Q) $\left|\vec{a}_1\times\vec{a}_2 + \vec{a}_2\times\vec{a}_3 + \ldots + \vec{a}_{n-1}\times\vec{a}_n\right|$

$= \left|\vec{a}_1 . \vec{a}_2 + \vec{a}_2 . \vec{a}_3 + \ldots + \vec{a}_{n-1} . \vec{a}_n\right|$

Let $= |\vec{a}_1| = |\vec{a}_2|$ ..... $= |\vec{a}_n| = \lambda$ (as centre is origin)

More over angle between 2 consecutive $\vec{a}_i$'s is $\frac{2\pi}{n}$

Hence given equation reduces to

$(n-1)\lambda^2 \sin\left(\frac{2\pi}{n}\right) = (n-1)\lambda^2\cos\left(\frac{2\pi}{n}\right)$

$\Rightarrow$ $\tan\left(\frac{2\pi}{n}\right) = 1$ $\Rightarrow$ $\frac{2\pi}{n} = \frac{\pi}{4}$ $\Rightarrow$ $n = 8$

(R) Equation of normal $\frac{6x}{h} - \frac{3y}{1} = 3$ $\left(\text{Equation of normal is } \frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2\right)$

slope = $\frac{6}{3h}$ = 1 (as it is perpendicular to z + y = 1) $\Rightarrow$ h = 2

(S) $\tan^{-1}\left(\frac{1}{2x+1}\right) + \tan^{-1}\left(\frac{1}{4x+1}\right) + \tan^{-1}\left(\frac{2}{x^2}\right)$

$\Rightarrow \frac{\frac{1}{2x+1} + \frac{1}{4x+1}}{1 - \frac{1}{(2x+1)(4x+1)}} = \frac{2}{x^2}$ $\Rightarrow$ $\frac{6x+2}{8x^2+6x} = \frac{2}{x^2}$

$\Rightarrow$ $3x^3 + x^2 = 8x^2 + 6x$ $\Rightarrow$ $3x^3 - 7x^2 - 6x = 0$

$\Rightarrow$ $3x^2 - 7x + 6 = 0$ (as $x \neq 0$)

$\Rightarrow$ $(x - 3)(3x + 2) = 0$ $\Rightarrow$ $x = -\frac{2}{3}, 3$ $\left(-\frac{2}{3} \text{ is rejected}\right)$

Hindi (P) $y = 4x^3 - 3x$ जहाँ $\cos\theta = x$

$\frac{dy}{dx} = 12x^2 - 3$

$\frac{d^2y}{dx^2} + x\frac{dy}{dx} = (x^2 - 1) . 24x + x(12x^2 - 3)$

$= 36x^3 - 27x = 9(4x^3 - 3x) = 9y$

अतः $\frac{1}{y}\left\{(x^2-1)\frac{d^2y}{dx^2} + x\frac{dy}{dx}\right\} = 9$

(Q) $\left|\vec{a}_1 \times \vec{a}_2 + \vec{a}_2 \times \vec{a}_3 + \ldots + \vec{a}_{n-1} \times \vec{a}_n\right|$

$= \left|\vec{a}_1 . \vec{a}_2 + \vec{a}_2 . \vec{a}_3 + \ldots + \vec{a}_{n-1} . \vec{a}_n\right|$

माना $= |\vec{a}_1| = |\vec{a}_2| \; \ldots\ldots = |\vec{a}_n| = \lambda$ (चूंकि केन्द्र, मूल बिन्दु है)

दो क्रमागत $\vec{a}_i$'s के मध्य कोण $\frac{2\pi}{n}$ है

अतः दिए गए समीकरण होती है

$(n - 1)\lambda^2 \sin\left(\frac{2\pi}{n}\right) = (n - 1)\lambda^2 \cos\left(\frac{2\pi}{n}\right)$

$\Rightarrow$ $\tan\left(\frac{2\pi}{n}\right) = 1$ $\Rightarrow$ $\frac{2\pi}{n} = \frac{\pi}{4}$ $\Rightarrow$ n = 8

(R) अभिलम्ब का समीकरण $\frac{6x}{h} - \frac{3y}{1} = 3$ $\left(\text{Equation of normal is } \frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2\right)$

प्रवणता $= \frac{6}{3h}$ = 1 (चूंकि यह z + y = 1 के लम्बवत् है) $\Rightarrow$ h = 2

(S) $\tan^{-1}\left(\frac{1}{2x+1}\right) + \tan^{-1}\left(\frac{1}{4x+1}\right) + \tan^{-1}\left(\frac{2}{x^2}\right)$

$$\Rightarrow \frac{\frac{1}{2x+1}+\frac{1}{4x+1}}{1-\frac{1}{(2x+1)(4x+1)}} = \frac{2}{x^2} \qquad \Rightarrow \qquad \frac{6x+2}{8x^2+6x} = \frac{2}{x^2}$$

$\Rightarrow \quad 3x^3 + x^2 = 8x^2 + 6x \quad \Rightarrow \quad 3x^3 - 7x^2 - 6x = 0$

$\Rightarrow \quad 3x^2 - 7x + 6 = 0$ (चूंकि $x \neq 0$)

$\Rightarrow \quad (x - 3)(3x + 2) = 0 \quad \Rightarrow \quad x = -\frac{2}{3}, 3 \qquad \left(-\frac{2}{3} \text{ is rejected}\right)$

29*. In R^3, consider the planes $P_1 : y = 0$ and $P_2 : x + z = 1$. Let P_3 be a plane, different from P_1 and P_2, which passes through the intersection of P_1 and P_2. If the distance of the point (0, 1, 0) from P_3 is 1 and the distance of a point (α, β, γ) from P_3 is 2, then which of the following relation is (are) true ?

मान लीजिए कि R^3 में $P_1 : y = 0$ और $P_2 : x + z = 1$ दो समतल है। माना कि P_3 एक समतल है जो समतल P_1 एवं P_2 से भिन्न है तथा P_1 एवं P_2 के प्रतिच्छेदन (intersection) से जाता है। यदि बिन्दु (0, 1, 0) से P_3 की दूरी एक (1) है तथा बिन्दु (α, β, γ) से P_3 की दूरी दो (2) है, तब निम्नलिखित सम्बंध (सम्बंधों) में कौन सा (से) संतुष्टित होते है। (हैं) ?

[JEE (Advanced) 2015, P-1 (4, –2)/ 88]

(A) $2\alpha + \beta + 2\gamma + 2 = 0$ (B*) $2\alpha - \beta + 2\gamma + 4 = 0$

(C) $2\alpha + \beta - 2\gamma - 10 = 0$ (D*) $2\alpha - \beta + 2\gamma - 8 = 0$

Ans. **(B,D)**

Sol. Let P_3 be $P_2 + \lambda P_1 = 0 \quad \Rightarrow \quad x + \lambda y + z - 1 = 0$

Distance from (0, 1, 0) is 1

$$\therefore \quad \frac{0+\lambda+0-1}{\sqrt{1+\lambda^2+1}} = \pm 1$$

$$\lambda = -\frac{1}{2}$$

$\therefore$ Equation of P_3 is $2x - y + 2z - 2 = 0$

Dist. from (α, β, γ) is 3

$$\therefore \quad \left|\frac{2\alpha-\beta+2\gamma-2}{3}\right| = 2 \quad \Rightarrow \quad 2\alpha - \beta + 2\gamma = 2 \pm 6$$

$\therefore$ option (B, D) are correct.

30*. In R^3, let L be a straight line passing through the origin. Suppose that all the points on L are at a constant distance from the two planes $P_1 : x + 2y - z + 1 = 0$ and $P_2 : 2x - y + z - 1 = 0$. Let M be the locus of the feet of the perpendiculars drawn from the points on L to the plane P_1. Which of the following points lie(s) on M?

माना कि R^3 में L एक सरल रेखा है जो कि मूल बिन्दु से जाती है। माना कि L के सभी बिन्दु समतलों $P_1 : x + 2y - z + 1 = 0$ तथा $P_2 : 2x - y + z - 1 = 0$ से स्थिर दूरी पर है। माना कि L के बिन्दुओं से समतल P_1 पर डाले गए लम्बों के पादों (feet of the perpendiculars) का पथ (locus) M है। निम्नलिखित बिन्दुओं मे से कौन सा (से) बिन्दु पथ M पर स्थित है है (है) ? **[JEE (Advanced) 2015, P-1 (4, –2)/ 88]**

(A*) $\left(0, -\frac{5}{6}, -\frac{2}{3}\right)$ (B*) $\left(-\frac{1}{6}, -\frac{1}{3}, \frac{1}{6}\right)$ (C) $\left(-\frac{5}{6}, 0, \frac{1}{6}\right)$ (D) $\left(-\frac{1}{3}, 0, \frac{2}{3}\right)$

Ans. **(A,B)**

Sol. Let $\vec{v}$ be the vector along L

$$\text{then } \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{vmatrix} = \hat{i} - 3\hat{j} - 5\hat{k}$$

So any point on line L is A(λ, – 3λ, –5λ)
Foot of perpendicular from A to P, is

$$\frac{h-\lambda}{1}=\frac{k-3\lambda}{2}=\frac{\ell+5\lambda}{-1}=-\frac{(\lambda-6\lambda+5\lambda+1)}{1+4+1}=-\frac{1}{6}$$

$$h=\lambda-\frac{1}{6},\ k=-3\lambda-\frac{1}{3},\ \ell=-5\lambda+\frac{1}{6}$$

so foot is $\left(\lambda-\frac{1}{6},-3\lambda-\frac{1}{3},-5\lambda+\frac{1}{6}\right)$

So (A, B)

Hindi. माना L के अनुदिश सदिश $\vec{v}$ L

तब $\vec{v}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & -1\\ 2 & -1 & 1\end{vmatrix}=\hat{i}-3\hat{j}-5\hat{k}$

अतः L पर कोई बिन्दु A(λ, – 3α, –5α) है
A से P पर लम्बपाद

$$\frac{h-\lambda}{1}=\frac{k-3\lambda}{2}=\frac{\ell+5\lambda}{-1}=-\frac{(\lambda-6\lambda+5\lambda+1)}{1+4+1}=-\frac{1}{6}$$

$$h=\lambda-\frac{1}{6},\ k=-3\lambda-\frac{1}{3},\ \ell=-5\lambda+\frac{1}{6}$$

अतः लम्बपाद $\left(\lambda-\frac{1}{6},-3\lambda-\frac{1}{3},-5\lambda+\frac{1}{6}\right)$ है। अतः (A, B) विकल्प सही है।

31*. Let ΔPQR be a triangle. Let $\vec{a}=\overrightarrow{QR},\vec{b}=\overrightarrow{RP}$ and $\vec{c}=\overrightarrow{PQ}$. If $|\vec{a}|$ = 12, $|\vec{b}|=4\sqrt{3}$ and $\vec{b}.\vec{c}=24$, then which of the following is(are) true?

(A*) $\frac{|\vec{c}|^2}{2}-|\vec{a}|=12$ (B) $\frac{|\vec{c}|^2}{2}+|\vec{a}|=30$

(C*) $|\vec{a}\times\vec{b}+\vec{c}\times\vec{a}|=48\sqrt{3}$ (D*) $\vec{a}.\vec{b}=-72$

माना कि ΔPQR एक त्रिभुज है। माना कि $\vec{a}=\overrightarrow{QR},\vec{b}=\overrightarrow{RP}$ और $\vec{c}=\overrightarrow{PQ}$ है। यदि $|\vec{a}|$ = 12 $|\vec{b}|=4\sqrt{3}$ और $\vec{b}.\vec{c}=24$, तब निम्नलिखित में से कौन सा (से) सही है (हैं)? **[JEE (Advanced) 2015, P-1 (4, –2)/ 88]**

(A) $\frac{|\vec{c}|^2}{2}-|\vec{a}|=12$ (B) $\frac{|\vec{c}|^2}{2}+|\vec{a}|=30$

(C) $|\vec{a}\times\vec{b}+\vec{c}\times\vec{a}|=48\sqrt{3}$ (D) $\vec{a}.\vec{b}=-72$

Ans. **(A,C,D)**

Sol. $\vec{a}+\vec{b}+\vec{c}=0$

⇒ $\vec{b}+\vec{c}=-\vec{a}$

⇒ 48 + + 48 = 144

⇒ $\vec{c}^2=48$

⇒ $|\vec{c}^2|=4\sqrt{3}$

∴ $\frac{|\vec{c}|^2}{2}-|\vec{a}|$ = 24 – 12 = 12 **Ans. (A)**

Further यद्यपि

$\vec{a}+\vec{b}=-\vec{c}$

$\Rightarrow$ $144 + 48 + 2\vec{a}.\vec{b} = 48$

$\Rightarrow$ $\vec{a}.\vec{b} = -72$ **Ans. (D)**

$\because$ $\vec{a}+\vec{b}+\vec{c} = 0$

$\Rightarrow$ $\vec{a}\times\vec{b}+\vec{a}\times\vec{c} = 0$

$\therefore$ $\left|\vec{a}\times\vec{b}+\vec{c}\times\vec{a}\right| = 2 = 2.\sqrt{144.48-(72)^2} = 48\sqrt{3}$ **Ans. (C)**

32. **Column- I** **Column-II**

(A) In R^2, if the magnitude of the projection vector of the vector $\alpha\hat{i}+\beta\hat{j}$ on $\sqrt{3}\hat{i}+\hat{j}$ is $\sqrt{3}$ and if $\alpha = 2 + \sqrt{3}\ \beta$, then possible value(s) of $|\alpha|$ is (are) (P) 1

(B) Let a and b be real numbers such that the function $f(x)=\begin{cases}-3ax^2-2, & x<1\\ bx+a^2, & x\geq 1\end{cases}$ is differentiable for all $x \in R$. Then possible value(s) of a is (are) (Q) 2

(C) Let $\omega \neq 1$ be a complex cube root of unity. If $(3-3\omega+2\omega^2)^{4n+3} + (2+3\omega-3\omega^2)^{4n+3} + (-3+2\omega+3\omega^2)^{4n+3} = 0$, then possible value(s) of n is (are) (R) 3

(D) Let the harmonic mean of two positive real numbers a and b be 4. If q is a positive real number such that a, 5,q, b is an arithmetic progression, then the value(s) of $|q-a|$ is (are) (S) 4

(T) 5

[JEE (Advanced) 2015, P-1 (2, –1)/ 88]

कॉलम - I **कॉलम-II**

(A) माना कि R^2 में, यदि सदिश $\alpha\hat{i}+\beta\hat{j}$ का सदिश $\sqrt{3}\hat{i}+\hat{j}$ पर प्रक्षेप सदिश (projection vector) का परिमाण (magnitude) $\sqrt{3}$ हो और यदि $\alpha = 2 + \sqrt{3}\ \beta$ हो, तब $|\alpha|$ के संभव मान है (हैं) (P) 1

(B) माना कि वास्तविक संख्याएं a और b इस प्रकार है कि फलन $f(x)=\begin{cases}-3ax^2-2, & x<1\\ bx+a^2, & x\geq 1\end{cases}$ सभी $x \in R$ के लिए अवकलनीय है। तब α के संभव मान है (हैं) (Q) 2

(C) माना कि $\omega \neq 1$, इकाई (unity) का एक सम्मिश्र घनमूल है। यदि $(3-3\omega+2\omega^2)^{4n+3} + (2+3\omega-3\omega^2)^{4n+3} + (-3+2\omega+3\omega^2)^{4n+3} = 0$, तब n के संभव मान है (हैं) (R) 3

(D) माना कि दो धनात्मक वास्तविक संख्याएं a और b का हरात्मक माध्य 4 है। यदि एक धनात्मक वास्तविक संख्या q इस प्रकार है कि a, 5,q, b एक समान्तर श्रेणी है। तब $|q-a|$ का (के) मान है (हैं) (S) 4

[JEE (Advanced) 2015, P-1 (2, –1)/ 88]

Ans. **(A) → P,Q ; (B) → P, Q ; (C) → P,Q,S,T ; (D) → Q, T**

Sol. (A) $\left|(\alpha\hat{i}+\beta\hat{j}).\left(\frac{\sqrt{3}\hat{i}+\hat{j}}{2}\right)\right| = \sqrt{3}$ $\Rightarrow$ $\sqrt{3}\alpha+\beta = \pm 2\sqrt{3}$

$\sqrt{3}\,\alpha + \left(\frac{\alpha-2}{\sqrt{3}}\right) = \pm 2\sqrt{3}$

$\Rightarrow$ $3\alpha + \alpha - 2 = \pm 6$ $\Rightarrow$ $4\alpha = 8, -4$ $\Rightarrow$ $\alpha = 2, -1$

(A → P, Q)

(B) Continuous सतत् $\Rightarrow$ $-3a - 2 = b + a^2$

differentiable अवकलनीय $\Rightarrow$ $-6a = b \Rightarrow$ $6a = a^2 + 3a + 2$

$\Rightarrow$ $a^2 - 3a + 2 = 0$ $\Rightarrow$ $a = 1,2$

(B → P, Q)

(D) $\frac{2ab}{a+b} = 4$ $\Rightarrow$ $ab = 2a + 2b$ ......(i)

$q = 10 - a$ and तथा $2q = 5 + b$

$\Rightarrow$ $20 - 2a = 5 + b$ $\Rightarrow$ $15 = 2a + b$ ....(ii)

From (i) and और (ii) से $a\,(15 - 2a) = 2a + 2(15 - 2a)$

$\Rightarrow$ $15a - 2a^2 = -2a + 30$ $\Rightarrow$ $2a^2 - 17a + 30 = 0$ $\Rightarrow$ $a = 6, \frac{5}{2}$

$\Rightarrow$ $q = 4, \frac{5}{2}$ $\Rightarrow$ $|q - a| = 2, 5$

(D → Q, T)

(C) Let $a = 3 - 3\omega + 2\omega^2$

$a\omega = 3\omega - 3\omega^2 + 2$

$a\omega^2 = 3\omega^2 - 3 + 2\omega$

Now $a^{4n+3}\,(1 + \omega^{4x+3} + (\omega^2)^{4n+3}) = 0$

$\Rightarrow$ n should not be a multiple of 3

Hence P, Q, S, T

33. **Column- I** **Column-II**

(A) In a triangle ΔXYZ, let a, b and c be the lengths of the sides opposite to the angles X, Y and Z, respectively. If $2(a^2 - b^2) = c^2$ and $\lambda = \frac{\sin(X-Y)}{\sin Z}$, then possible values of n for which $\cos(n\pi\lambda) = 0$ is (are) — (P) 1

(B) In a triangle ΔXYZ, let a, b and c be the lengths of the sides opposite to the angles X, Y and Z, respectively. If $1 + \cos 2X - 2\cos 2Y = 2\sin X \sin Y$, then possible value(s) of $\frac{a}{b}$ is (are) — (Q) 2

(C) In R^2, let $\sqrt{3}\,\hat{i}+\hat{j}$, $\hat{i}+\sqrt{3}\hat{j}$ and $\beta\ \hat{i}+(1-\beta)\ \hat{j}$ be the position vectors of X, Y and Z with respect to the origin O, respectively. If the distance of Z from the bisector of the acute angle of $\overrightarrow{OX}$ and $\overrightarrow{OY}$ — (R) 3 is , $\dfrac{3}{\sqrt{2}}$ then possible value(s) of $|\beta|$ is (are)

(D) Suppose that $F(\alpha)$ denotes the area of the region bounded by $x = 0$, $x = 2$, $y^2 = 4x$ and $y = |\alpha x - 1| + |\alpha x - 2| + \alpha x$, where $\alpha \in \{0, 1\}$. Then the value(s) of $F(\alpha) + \dfrac{8}{3}\sqrt{2}$, when $\alpha = 0$ and $\alpha = 1$, is (are) — (S) 5

(T) 6

[JEE (Advanced) 2015, P-1 (2, –1)/ 88]

कॉलम - I — **कॉलम-II**

(A) माना कि एक त्रिभुज ΔXYZ में कोणों X, Y और Z के सामने की भुजाओं की लम्बाईयाँ क्रमशः a, b और c है। मानाकि $2(a^2 - b^2) = c^2$ और $\lambda = \dfrac{\sin(X-Y)}{\sin Z}$ है। यदि $\cos(n\pi\lambda) = 0$ तब n के संभव मान है (हैं) — (P) 1

(B) माना कि एक त्रिभुज ΔXYZ में कोणों X, Y और Z के सामने की भुजाओं की लम्बाईयाँ क्रमशः a, b और c है। यदि $1 + \cos 2X - 2\cos 2Y = 2\sin X \sin Y$ तब $\dfrac{a}{b}$ के संभव मान है (हैं) — (Q) 2

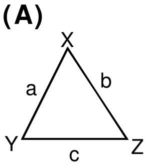
(C) माना कि R^2 में, मूल बिन्दु O के सापेक्ष $\sqrt{3}\,\hat{i}+\hat{j},\ \hat{i}+\sqrt{3}\hat{j}$ और $\beta\ \hat{i}+(1-\beta)\ \hat{j}$ क्रमशः X, Y और Z के स्थिति सदिश (position vectors) है। यदि $\overrightarrow{OX}$ और $\overrightarrow{OY}$ के न्यून कोण के द्विभाजक से Z की दूरी $\dfrac{3}{\sqrt{2}}$ हो, तो $|\beta|$ का (के) संभव मान है (हैं) — (R) 3

(D) माना कि $F(\alpha)$ उस क्षेत्र के क्षेत्रफल को दर्शाता है जो $x = 0$, $x = 2$, $y^2 = 4x$ और $y = |\alpha x - 1| + |\alpha x - 2| + \alpha x$, से घिरा है, जहाँ $\alpha \in \{0, 1\}$ है। $\alpha = 0$ और $\alpha = 1$ के लिए $F(\alpha) + \dfrac{8}{3}\sqrt{2}$ का (के) मान है (हैं) — (S) 5

(T) 6

Ans. **(A) → P,R,S ; (B) → P ; (C) → P,Q ; (D) → S, T**

Sol. **(A)**



Given $2(a^2 - b^2) = c^2$

$\Rightarrow \quad 2(\sin^2 x - \sin^2 y) = \sin^2 z$

$\Rightarrow \quad 2\sin(x + y)\sin(x - y) = \sin^2 z$

$\Rightarrow \quad 2\sin(\pi - z)\sin(x - y) = \sin^2 z$

z $\quad \Rightarrow \quad \sin(x - y) = \dfrac{\sin z}{2} \quad \ldots(i)$

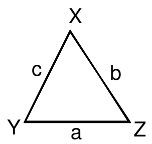
also given,

$\lambda = \dfrac{\sin(x-y)}{\sin z} = \dfrac{1}{2}$

Now, $\cos(n\pi\lambda) = 0$

$\Rightarrow \quad \cos\left(\dfrac{n\pi}{2}\right) = 0$

$\therefore \quad n = 1, 3, 5 \qquad \therefore \quad (A \to P,R,S)$



(B)

$1 + \cos 2X - 2\cos 2Y = 2\sin X \sin Y$

$2\cos^2 X - 2\cos 2Y = 2\sin X \sin Y$

$1 - \sin^2 X - 1 + 2\sin^2 Y = \sin X \sin Y$

$\sin^2 X + \sin X \sin Y = 2\sin^2 Y$

$\sin(\sin X + \sin Y) = 2\sin^2 Y \qquad \sin X = ak,\ \sin Y = bk$

$a(a + b) = 2b^2$

$a^2 + ab - 2b^2 = 0$

$\left(\dfrac{a}{b}\right)^2 + \dfrac{a}{b} - 2 = 0$

$\dfrac{a}{b} = -2, 1$

$\dfrac{a}{b} = 1 \qquad$ **(B → P)**

Hence equation of acute angle bisector of OX and OY is y = x

Hence x – y = 0

Now, distance of $\beta\hat{i} + (1-\beta)\hat{j} \equiv z(\beta, 1 - \beta)$ from x – y is $\left|\dfrac{\beta-(1-\beta)}{\sqrt{2}}\right| = \dfrac{3}{\sqrt{2}}$

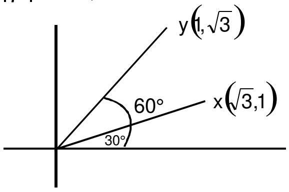
$|2\beta - 1| = 3$

$2\beta - 1 = \pm 3$

$2\beta = 4, -2$

$\beta = 2, -1$

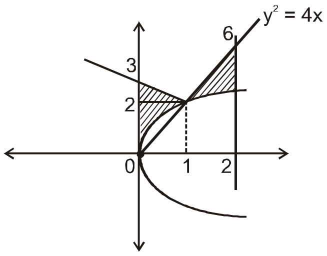
$|\beta| = 2, 1 \qquad$ Ans. (P,Q)



(D)

For $\alpha = 1$

$y = |x-1| + |x-2| + x = \begin{cases} 3-x & ; \quad x < 1 \\ 1+x & ; \quad 1 \le x < 2 \\ 3x-3 & ; \quad x \ge 2 \end{cases}$

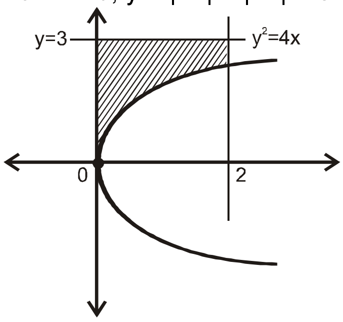


$$A = \frac{1}{2}(2+3)\times 1 + \frac{1}{2}(2+3)\times 1 - \int_0^2 2\sqrt{x}dx$$

$$A = 5 - \frac{8}{3}\sqrt{2}$$

$\therefore \quad F(1) + \frac{8}{3}\sqrt{2} = 5$

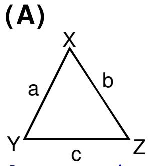
For $\alpha = 0$, $y = |-1| + |-2| = 3$



$$A = 6 - \int_0^2 2\sqrt{x}dx \quad \Rightarrow \quad A = 6 - \frac{8}{3}\sqrt{2}$$

$\therefore \quad F(0) + \frac{8}{3}\sqrt{2} = 6 \qquad \therefore \quad (D \to s, t)$

Hindi. (A)



दिया गया है $2(a^2 - b^2) = c^2$

$\Rightarrow \quad 2(\sin^2 x - \sin^2 y) = \sin^2 z$

$\Rightarrow \quad 2\sin(x + y)\sin(x - y) = \sin^2 z$

$\Rightarrow \quad 2\sin(\pi - z)\sin(x - y) = \sin^2 z$

$z \quad \Rightarrow \quad \sin(x - y) = \frac{\sin z}{2} \quad \ldots(i)$

यह भी ज्ञात है,

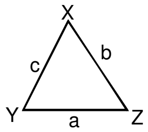
$$\lambda = \frac{\sin(x-y)}{\sin z} = \frac{1}{2}$$

अब , $\cos(n\pi\lambda) = 0$

$\Rightarrow \quad \cos\left(\frac{n\pi}{2}\right) = 0$

$\therefore \quad n = 1, 3, 5 \qquad \therefore \quad (Q \to P,R,S)$

(B)



$1 + \cos 2x - 2\cos 2y = 2\sin x \sin y$
$2\cos^2 x - 2\cos 2y = 2\sin x \sin y$
$1 - \sin^2 x - 1 + 2\sin^2 y = \sin x \sin y$
$\sin^2 x + \sin x \sin y = 2\sin^2 y$
$\sin(\sin x + \sin y) = 2\sin^2 y \qquad \sin x = ak,\ \sin y = bk$
$a(a + b) = 2b^2$
$a^2 + ab - 2b^2 = 0$

$1 + \cos 2x - 2\cos 2y = 2\sin x \sin y$
$2\cos^2 x - 2\cos 2y = 2\sin x \sin y$
$1 - \sin^2 x - 1 + 2\sin^2 y = \sin x \sin y$
$\sin^2 x + \sin x \sin y = 2\sin^2 y$
$\sin(\sin x + \sin y) = 2\sin^2 y \qquad \sin x = ak,\ \sin y = bk$
$a(a + b) = 2b^2$
$a^2 + ab - 2b^2 = 0$

$\left(\frac{a}{b}\right)^2 + \frac{a}{b} - 2 = 0$

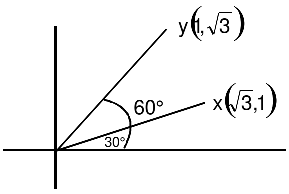
$\frac{a}{b} = -2, 1$

$\frac{a}{b} = 1 \qquad$ **(B → P)**

(C)

अतः OX तथा OY के मध्य न्यून कोण के अर्धक का समीकरण $y = x$ है।
अतः $x - y = 0$

अब, $\beta\hat{i} + (1-\beta)\hat{j} \equiv z(\beta, 1 - \beta)$ की $\qquad x - y$ से दूरी $\left|\frac{\beta - (1-\beta)}{\sqrt{2}}\right| = \frac{3}{\sqrt{2}}$

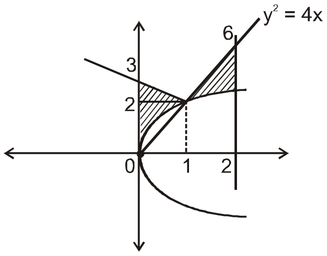


$|2\beta - 1| = 3$
$2\beta - 1 = \pm 3$
$2\beta = 4, -2$
$\beta = 2, -1$

$|\beta| = 2, 1 \qquad$ Ans. (P,Q)

(D)

For $\alpha = 1 \qquad y = |x - 1| + |x - 2| + x = \begin{cases} 3 - x & ; \quad x < 1 \\ 1 + x & ; \quad 1 \le x < 2 \\ 3x - 3 & ; \quad x \ge 2 \end{cases}$

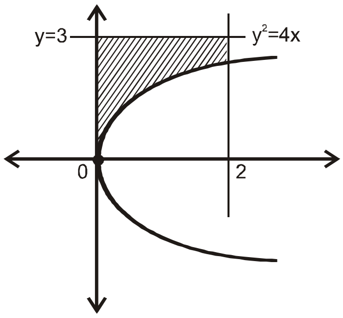


$$A = \frac{1}{2}(2+3)\times 1 + \frac{1}{2}(2+3)\times 1 - \int_0^2 2\sqrt{x}dx$$

$$A = 5 - \frac{8}{3}\sqrt{2}$$

$\therefore$ F(1) + = 5

के लिए $\alpha = 0$, $y = |-1| + |-2| = 3$



$$A = 6 - \int_0^2 2\sqrt{x}dx \quad \Rightarrow \quad A = 6 - \frac{8}{3}\sqrt{2}$$

$\therefore$ F(0) + $\frac{8}{3}\sqrt{2}$ = 6 $\qquad \therefore$ (D $\to$ s, t)

34. Suppose that $\vec{p}, \vec{q}$ and $\vec{r}$ are three non-coplanar vectors in R^3. Let the components of a vector $\vec{s}$ along $\vec{p}, \vec{q}$ and $\vec{r}$ be 4, 3 and 5, respectively. If the components of this vector $\vec{s}$ along $(-\vec{p}+\vec{q}+\vec{r}), (\vec{p}-\vec{q}+\vec{r})$ and $(-\vec{p}-\vec{q}+\vec{r})$ are x, y and z, respectively, then the value of 2x + y + z is

माना कि R^3 में, $\vec{p}, \vec{q}$ और $\vec{r}$ तीन असमतलीय सदिश है। माना कि सदिश के घटक क्रमागत सदिशों $\vec{p}, \vec{q}$ तथा $\vec{r}$ के अनुदिश क्रमशः 4, 3 तथा 5 है। यदि के घटक क्रमागत सदिशों $(-\vec{p}+\vec{q}+\vec{r}), (\vec{p}-\vec{q}+\vec{r})$ एवं $(-\vec{p}-\vec{q}+\vec{r})$ के अनुदिश क्रमशः x, y तथा z है, तब 2x + y + z का मान है। **[JEE (Advanced) 2015, P-2 (4, 0) / 80]**

Ans. **BONUS**

Sol. ***This question in seem to be wrong but examiner may think like this***

यह प्रश्न गलत लग रहा है परन्तु परीक्षक इस प्रकार सोच सकता है।

Ans. **BONUS**

Sol. ***This question in seem to be wrong but examiner may think like this***

यह प्रश्न गलत लग रहा है परन्तु परीक्षक इस प्रकार सोच सकता है।

$$\vec{S} = 4\vec{p} + 3\vec{q} + 5\vec{r}$$

$$\vec{S} = x(-\vec{p}+\vec{q}+\vec{r}) + y(\vec{p}-\vec{q}+\vec{r}) + z(-\vec{p}-\vec{q}+\vec{r})$$

$-x + y - z = 4$ .....(1)
$x - y - z = 3$ .....(2)
$x + y + z = 5$ .....(3)
add (1) and (2)
(1) और (2) को जोड़ने पर

$-2z = 7 \Rightarrow \qquad z = -\frac{7}{2}$

$2x = 8 \Rightarrow \qquad x = 4$

$y + z = 1$

$2x + y + z = 2(4) + 1 = 9$

35*. Consider a pyramid OPQRS located in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$) with O as origin, and OP and OR along the x-axis and the y-axis, respectively. The base OPQR of the pyramid is a square with OP = 3. The point S is directly above the mid point T of diagonal OQ such that TS = 3. Then

(A) the acute angle between OQ and OS is $\frac{\pi}{3}$ **[JEE (Advanced) 2016, Paper-1, (4, –2)/62]**

(B*) the equation of the plane containing the triangle OQS is $x - y = 0$

(C*) the length of the perpendicular from P to the plane containing the triangle OQS is $\frac{3}{\sqrt{2}}$

(D*) the perpendicular distance from O to the straight line containing RS is $\sqrt{\frac{15}{2}}$

विचार कीजिये, एक सूच्याकार (pyramid) OPQRS जो प्रथम अष्टांशक (first octant) ($x \geq 0$, $y \geq 0$, $z \geq 0$) में स्थित है, जिसमें O मूलबिन्दु (origin) तथा OP और OR क्रमशः x- अक्ष और y- अक्ष पर है। इस सूच्याकार का आधार (base) OPQR एक वर्ग (square) है जिसमें OP = 3 है। बिन्दु S कर्ण (diagonal) OQ के मध्यबिन्दु T के ठीक ऊपर इस प्रकार है कि TS = 3 है, तब

(A) OQ और OS के बीच का न्यूनकोण (acute angle) $\frac{\pi}{3}$ है

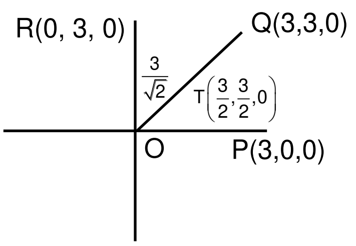
(B) त्रिभुज OQS को अंतर्विष्ट (contain) करने वाले समतल का समीकरण $x - y = 0$ है

(C) P से त्रिभुज OQS को अंतर्विष्ट करने वाले समतल पर लम्ब की लम्बाई $\frac{3}{\sqrt{2}}$ है

(D) O से RS को अंतर्विष्ट करती हुई सरल रेखा की लम्बवत् दूरी $\sqrt{\frac{15}{2}}$ है।

Ans. **(B,C,D)**

Sol.



$S \equiv \left(\frac{3}{2}, \frac{3}{2}, 3\right) \Rightarrow \qquad \overrightarrow{OQ} = 3\hat{i} + 3\hat{j} \quad \Rightarrow \qquad \overrightarrow{OS} = \frac{3}{2}\hat{i} + \frac{3}{2}\hat{j} + 3\hat{k}$

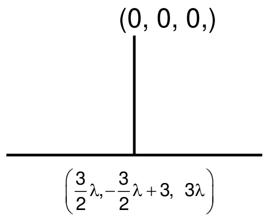
$$\cos\theta = \frac{\frac{1}{2}+\frac{1}{2}}{\sqrt{2}\sqrt{\frac{1}{2}+\frac{1}{4}+1}} = \frac{1}{\sqrt{2}\sqrt{\frac{3}{2}}} = \frac{1}{\sqrt{3}}$$

$$\vec{n} = \overrightarrow{OQ}\times\overrightarrow{OS} = (\hat{i}+\hat{j})\times(\hat{i}+\hat{j}+2\hat{k}) = \hat{k}-2\hat{j}-\hat{k}+2\hat{i} \quad\Rightarrow\quad 2\hat{i}-2\hat{j}$$

$$x - y = \lambda \quad\Rightarrow\quad x = y \quad\Rightarrow\quad \perp (3, 0, 0) \Rightarrow \frac{3}{\sqrt{2}}$$

$$RS \rightarrow \frac{x-0}{\frac{3}{2}} = \frac{y-3}{-\frac{3}{2}} = \frac{z-0}{3} = \lambda \quad\Rightarrow\quad x = \frac{3}{2}\lambda,\ y = -\frac{3}{2}\lambda + 3,\ z = 3\lambda$$

T distance (दूरी) $\Rightarrow \sqrt{\frac{3}{2}-3+9} \quad\Rightarrow\quad \sqrt{\frac{15}{2}}$



$$D = \frac{9}{4}\lambda^2 + \left(3-\frac{3}{2}\lambda\right)^2 + 9\lambda^2 = \frac{27}{2}\lambda^2 - 9\lambda + 9 \quad\Rightarrow\quad \lambda = \frac{9}{27} = \frac{1}{3}$$

36. Let P be the image of the point (3, 1, 7) with respect to the plane $x - y + z = 3$. Then the equation of the plane passing through P and containing the straight line $\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$ is

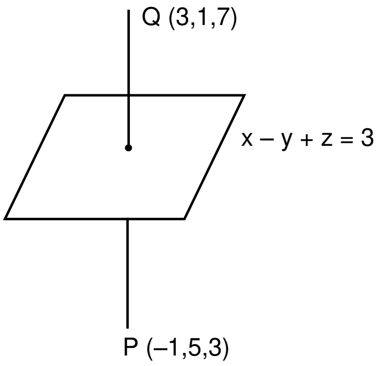
माना कि बिन्दु (3, 1, 7) का, समतल $x - y + z = 3$ के सापेक्ष (with respect to), प्रतिबिम्ब (image) P है। तब बिन्दु P से गुजरने वाले और सरल रेखा $\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$ को धारण करने वाले समतल का समीकरण है–

[JEE (Advanced) 2016, Paper-2, (3, –1)/62]

(A) $x + y - 3z = 0$ (B) $3x + z = 0$ (C) $x - 4y + 7z = 0$ (D) $2x - y = 0$

Ans. (C)

Sol.



$$\frac{x-3}{1} = \frac{y-1}{-1} = \frac{z-7}{1} = \frac{-2(6)}{3} = -4$$

$x = -1,\ y = 5,\ z = 3$ P (–1, 5, 3)

$a(x + 1) + b(y - 5) + c\,(z - 3) = 0$

$a + 2b + c = 0$ ....(i)

$a - 5b - 3c = 0$

$$\frac{a}{-1} = \frac{b}{4} = \frac{c}{-7}$$

$-(x + 1) + 4\,(y - 5) - 7\,(z - 3) = 0$

$-x + 4y - 7z = 0$

$x - 4y + 7z = 0$

37*. Let $\hat{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ be a unit vector in R^3 and $\hat{w} = \frac{1}{\sqrt{6}}\ \hat{i} + \hat{j} + 2\hat{k}$. Given that there exists a vector $\vec{v}$ in R^3 such that $|\hat{u} \times \vec{v}\,| = 1$ and $\hat{w}.\ \hat{u} \times \vec{v}\ = 1$. Which of the following statements(s) is (are) correct?

(A) There is exactly one choice for $\vec{v}$

(B) There are infinitely many choices for such $\vec{v}$ **[JEE (Advanced) 2016, Paper-2, (4, –2)/62]**

(C) If $\hat{u}$ lies in the xy-plane then $|u_1| = |u_2|$

(D) If $\hat{u}$ lies in the xz-plane then $2|u_1| = |u_3|$

माना कि R^3 में $\hat{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ एक मात्रक सदिश (unit vector) है और $\hat{w} = \frac{1}{\sqrt{6}}\,\hat{i} + \hat{j} + 2\hat{k}$ है। दिया हुआ है कि R^3 में सदिश $\vec{v}$ का अस्तित्व इस प्रकार है कि $|\hat{u}\times\vec{v}| = 1$ और $\hat{w}.\ \hat{u}\times\vec{v} = 1$ है। निम्नलिखित में से कौन सा (से) कथन सही है (हैं)?

(A) इस प्रकार के $\vec{v}$ के लिए ठीक एक (exactly one) चयन संभव है।

(B) इस प्रकार के $\vec{v}$ के लिए अनन्त (infinitely) चयन संभव है।

(C) यदि $\hat{u}$ xy- समतल पर है तब $|u_1| = |u_2|$ है।

(D) यदि $\hat{u}$ xz-समतल पर है तब $2|u_1| = |u_3|$ है।

Ans. **(B,C)**

Sol. $|\hat{u}\times\vec{v}| = 1$

$|v| \sin\theta = 1$ $\quad\theta$ is angle between $\hat{u}\ \&\ \vec{v}$ $\quad(\theta,\ \hat{u}\ \&\ \vec{v}$ के मध्य कोण है।)

Also तथा, $\quad\hat{w}.(\hat{u}\times\vec{v}) = 1$

$|\hat{w}|\,|\hat{u}|\ |\vec{v}|\sin\theta\cos\alpha = 1$ $\quad\alpha$ is angle between $\hat{w}\ \&\ (\hat{u}\times\vec{v})$ $\quad(\alpha,\ \hat{w}$ और $(\hat{u}\times\vec{v})$ के मध्य कोण है।)

$1.1(1)\cos\alpha = 1$ $\quad\Rightarrow\quad \boxed{\alpha = 0^\circ} \quad\Rightarrow\quad \boxed{\hat{u}\times\vec{v} = \lambda\hat{w}}$ where जहाँ $\lambda > 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_x & v_y & v_z \end{vmatrix} = \frac{\lambda}{\sqrt{6}}\left(\hat{i}+\hat{j}+2\hat{k}\right) \quad (u_2v_z - u_3v_y)\,\hat{i} + (u_3v_x - u_1v_z)\,\hat{j} + (u_1v_y - u_2v_x)\,\hat{k} = \frac{\lambda}{\sqrt{6}}\left(\hat{i}+\hat{j}+2\hat{k}\right)$$

(B) $\quad\vec{v}$ is a vector such that $(\hat{u}\times\vec{v})$ is parallel to $\hat{w}$.

(B) $\quad\vec{v}$ इस प्रकार है कि $(\hat{u}\times\vec{v}),\hat{w}$ के समान्तर है।

(C) $\quad u_3 = 0 \Rightarrow u_2v_z = \frac{\lambda}{\sqrt{6}}\ \&\ -u_1v_z = \frac{\lambda}{\sqrt{6}} \quad\Rightarrow\quad |u_2| = |u_1|$

(D) $\quad|u_1| = 2|u_3| \qquad (\because\ u_2 = 0)$

38. Let O be the origin and let PQR be an arbitrary triangle. The point S is such that

$\overrightarrow{OP}\cdot\overrightarrow{OQ}+\overrightarrow{OR}\cdot\overrightarrow{OS}=\overrightarrow{OR}\cdot\overrightarrow{OP}+\overrightarrow{OQ}\cdot\overrightarrow{OS}=\overrightarrow{OQ}\cdot\overrightarrow{OR}+\overrightarrow{OP}\cdot\overrightarrow{OS}$

Then the triangle PQR has S as its **[JEE(Advanced) 2017, Paper-2,(3, –1)/61]**

(A) centroid (B) orthocenter

(C) incentre (D) circumcenter

माना कि O मूलबिन्दु (origin) है एवम् PQR एक स्वेच्छिक त्रिभुज (arbitrary triangle) है। बिन्दु S इस प्रकार है कि

$\overrightarrow{OP}\cdot\overrightarrow{OQ}+\overrightarrow{OR}\cdot\overrightarrow{OS}=\overrightarrow{OR}\cdot\overrightarrow{OP}+\overrightarrow{OQ}\cdot\overrightarrow{OS}=\overrightarrow{OQ}\cdot\overrightarrow{OR}+\overrightarrow{OP}\cdot\overrightarrow{OS}$

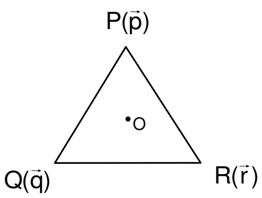
तब बिन्दु S त्रिभुज PQR का है

(A) केन्द्रक(centroid) (B) लम्बकेन्द्र (orthocenter)

(C) अन्तःकेन्द्र (incentre) (D) परिवृत्तकेन्द्र (circumcenter)

Ans. (B)

Sol.



$\vec{p}.\vec{q}+\vec{r}.\vec{s}=\vec{r}.\vec{p}+\vec{q}.\vec{s}=\vec{q}.\vec{r}+\vec{p}.\vec{s}$

$\Rightarrow \vec{p}.(\vec{q}-\vec{r})-\vec{s}.(\vec{q}-\vec{r})=0 \Rightarrow \overrightarrow{PS}.\overrightarrow{QR}=0$

Similarly इसी प्रकार $\overrightarrow{PQ}.\overrightarrow{SR}=0$

$\Rightarrow$ S is orthocentre of the triangle (S, त्रिभुज का लम्ब केन्द्र होगा)

39. The equation of the plane passing through the point (1, 1, 1) and perpendicular to the planes $2x + y - 2z = 5$ and $3x - 6y - 2z = 7$, is **[JEE(Advanced) 2017, Paper-2,(3, –1)/61]**

समतलों $2x + y - 2z = 5$ एवम् $3x - 6y - 2z = 7$ के लम्बवत् और बिन्दु (1, 1, 1) से गुजरने वाले समतल का समीकरण है

(A) $14x + 2y - 15z = 1$ (B) $-14x + 2y + 15z = 3$

(C) $14x - 2y + 15z = 27$ (D) $14x + 2y + 15z = 31$

Ans. **(D)**

Sol. Let plane be

$a(x - 1) + b(y - 1) + c(z - 1) = 0$

Now, direction ratio of its normal = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 3 & -6 & -2 \end{vmatrix} = \hat{i}(-14) - \hat{j}(2) + \hat{k}(-15)$

So, $-14(x - 1) - 2(y - 1) - 15(z - 1) = 0$

$14x + 2y + 15z = 31$

Hindi. **(D)**

माना समतल

$a(x - 1) + b(y - 1) + c(z - 1) = 0$

अभिलम्ब की द्विकोज्या = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 3 & -6 & -2 \end{vmatrix} = \hat{i}(-14) - \hat{j}(2) + \hat{k}(-15)$

अतः, $-14(x - 1) - 2(y - 1) - 15(z - 1) = 0$

$14x + 2y + 15z = 31$

Comprehension (Q.40 & 41)

Let O be the origin, and $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ be three unit vectors in the directions of the sides $\overrightarrow{QR}, \overrightarrow{RP}, \overrightarrow{PQ}$, respectively, of a triangle PQR.

अनुच्छेद (Q.40 & 41)

माना कि O मूलबिन्दु (origin) है एवम् $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ क्रमशः त्रिभुज PQR की भुजायें $\overrightarrow{QR}, \overrightarrow{RP}, \overrightarrow{PQ}$, की दिशाओ मे तीन एकक सदिश (unit vectors) है।

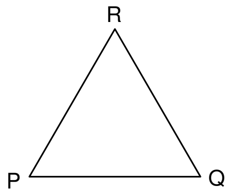
40. If the triangle PQR varies, then the minimum value of $\cos(P + Q) + \cos(Q + R) + \cos(R + P)$ is **[JEE(Advanced) 2017, Paper-2,(3, 0)/61]**

यदि त्रिभुज PQR परिवर्ती है (If the triangle PQR varies), तब, $\cos(P + Q) + \cos(Q + R) + \cos(R + P)$ का न्यूनतम मान (minimum value) है **Vector**

(A) $-\frac{3}{2}$ (B) $\frac{3}{2}$ (C) $\frac{5}{3}$ (D) $-\frac{5}{3}$

Ans. **(A)**

Sol.



$\cos(P + Q) + \cos(Q + R) + \cos(R + P) = -\cos R - \cos P - \cos Q$

In any किसी त्रिभुज में Δ, max of $\cos P + \cos Q + \cos R = \frac{3}{2}$

So minimum value of the given expression is $-\frac{3}{2}$

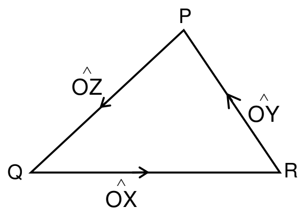
अतः दिए गए व्यंजक का न्यूनतम मान $-\frac{3}{2}$

41. $|\overrightarrow{OX} \times \overrightarrow{OY}| =$ **[JEE(Advanced) 2017, Paper-2,(3, 0)/61]**

(A) $\sin(P + Q)$ (B) $\sin(P + R)$ (C) $\sin(Q + R)$ (D) $\sin 2R$

Ans. **(A)** **Vector**

Sol.



$\cos R = -\ \hat{OX}.\hat{OY}$

$\Rightarrow |\cos R| = |\hat{OX}.\hat{OY}|$

$|\hat{OX} \times \hat{OY}| = |\sin R| = |\sin(\pi - (P + Q))| = |\sin(P + Q)| = \sin(P + Q)$

42*. Let $P_1 : 2x + y - z = 3$ and $P_2 : x + 2y + z = 2$ be two planes. Then, which of the following statement(s) is (are) TRUE? **[JEE(Advanced) 2018, Paper-1,(4, –2),60] [Vector]**

(A) The line of intersection of P_1 and P_2 has direction ratios 1, 2, – 1

(B) The line $\frac{3x-4}{9} = \frac{1-3y}{9} = \frac{z}{3}$ is perpendicular to the line of intersection of P_1 and P_2

(C) The acute angle between P_1 and P_2 is 60º

(D) If P_3 is the plane passing through the point (4, 2, –2) and perpendicular to the line of intersection of P_1 and P_2, then the distance of the point (2, 1, 1) from the plane P_3 is $\frac{2}{\sqrt{3}}$

मानाकि $P_1 : 2x + y - z = 3$ और $P_2 : x + 2y + z = 2$ दो समतल (plane) है। तब निम्नलिखित में से कौनसा (से) कथन सत्य है(हैं) ?

(A) P_1 और P_2 की प्रतिच्छेदन रेखा (line of intersection) के दिक्-अनुपात (direction ratios) 1, 2, – 1 है।

(B) रेखा $\frac{3x-4}{9} = \frac{1-3y}{9} = \frac{z}{3}$, P_1 और P_2 की प्रतिच्छेदन रेखा पर लम्बवत् (perpendicular) है।

(C) P_1 और P_2 के बीच का न्यून कोण (acute angle) 60º है।

(D) यदि समतल P_3, बिन्दु (4, 2, –2) से गुजरता है तथा P_1 और P_2 की प्रतिच्छेदन रेखा के लम्बवत् है, तब बिन्दु (2, 1, 1) की समतल P_3 से दूरी $\frac{2}{\sqrt{3}}$ है।

Ans. **(CD)**

Sol. Direction ratio of common line is $n_1 \times n_2$

उभनिष्ट रेखा के दि्क अनुपात $n_1 \times n_2$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = \hat{i}(3) - \hat{j}(3) + \hat{k}(3) = 3(\hat{i} - \hat{j} + \hat{k})$$

(B) $\frac{x-4/3}{3} = \frac{y-1/3}{-3} = \frac{z}{3}$

This is parallel to line of intersection

यह प्रतिच्छेदन रेखा के समान्तर है।

(C) $\cos\theta = \frac{x_i x_2}{|x_1||x_2|} = \frac{2+2-1}{\sqrt{6}\sqrt{6}} = \frac{3}{6} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

(D) $P_3 : x - y + z = \lambda$ satisfy संतुष्ट करता है (4, 2, –2)

$4 - 2 - 2 = \lambda \Rightarrow x - y + z = 0$

$(2,1,1) \perp \Rightarrow \left|\frac{2-1+1}{\sqrt{3}}\right| \Rightarrow \frac{2}{\sqrt{3}}$

43. Let $\vec{a}$ and $\vec{b}$ be two unit vectors such that $\vec{a}.\vec{b} = 0$. For some x, y $\in$ R, let $\vec{c} = x\vec{a} + y\vec{b} + (\vec{a} \times \vec{b})$. If $|\vec{c}| = 2$ and the vector $\vec{c}$ is inclined at the same angle α to both $\vec{a}$ and $\vec{b}$, then the value of $8\cos^2\alpha$ is _______. **[JEE(Advanced) 2018, Paper-1, (3, 0), 60] [Vector]**

मानाकि $\vec{a}$ और $\vec{b}$ दो ऐसे इकाई सदिश (unit vector) है कि $\vec{a}.\vec{b} = 0$. किन्ही x, y $\in$ R के लिए मानाकि $\vec{c} = x\vec{a} + y\vec{b} + (\vec{a} \times \vec{b})$.यदि $|\vec{c}| = 2$ और सदिश $\vec{c}$ सदिशों $\vec{a}$ और $\vec{b}$ दोनों के साथ समान कोण α बनाता है, तब $8\cos^2\alpha$ का मान है _______.

Ans. (3)

Sol. $\vec{c} = x\vec{a} + y\vec{b} + \vec{a}\times\vec{b}$ & $\vec{a}.\vec{b} = 0$

$\vec{a}\hat{\ }\vec{c} = \vec{b}\hat{\ }\vec{c} = \alpha$

$\vec{c}.\vec{a} = \vec{c}.\vec{b} = 2\cos\alpha \qquad \Rightarrow \qquad x = y = 2\cos\alpha$

$|\vec{c}|^2 = x^2 + y^2 + |\vec{a}\times\vec{b}|^2 = 2(4\cos^2\alpha) + 1 - 0$

$4 = 8\cos^2\alpha + 1 \qquad \Rightarrow \qquad 8\cos^2\alpha = 3$

44. Let P be a point in the first octant, whose image Q in the plane x + y = 3 (that is, the line segment PQ is perpendicular to the plane x + y = 3 and the mid-point of PQ lies in the plane x + y = 3) lies on the z-axis. Let the distance of P from the x-axis be 5. If R is the image of P in the xy-plane, then the length of PR is _______ . **[Vector] [JEE(Advanced) 2018, Paper-2,(3, 0)/60]**

माना कि P प्रथम अष्टांश (first octant) में एक बिंदु है, जिसका सममतल (plane) x + y = 3 में प्रतिबिम्ब (imgae) Q (अर्थात रेखाखण्ड PQ समतल x + y = 3 के लम्बवत है और PQ का मध्य बिंदु सममतल x + y = 3 में स्थित है) z-अक्ष (axis) पर स्थित है। माना कि P की x-अक्ष से दूरी 5 है। यदि P का xy-समतल में प्रतिबिम्ब R है, तब PR लम्बाई है _______

Ans. (8)

Sol. $P(\alpha, \beta, \gamma)$

$R(\alpha, \beta, -\gamma)$

Q

$$\frac{x-\alpha}{1} = \frac{y-\beta}{1} = \frac{z-\gamma}{0} = \frac{-2(\alpha+\beta-3)}{2}$$

$x = 3 - \beta,\ y = 3 - \alpha,\ z = \gamma$

$Q(3 - \beta, 3 - \alpha, \gamma)$ lies on z –axis

$Q(3 - \beta, 3 - \alpha, \gamma)$, z अ़क्ष पर स्थित है।

$\therefore \qquad \beta = 3, \alpha = 3$

$P(3, 3, \gamma)$ distance from x-axis is 5

$P(3, 3, \gamma)$, x-अक्ष से दूरी 5 है।

$9 + \gamma^2 = 25$

$\gamma^2 = 16 \qquad \Rightarrow \qquad \gamma = 4$

$P(3, 3, 4) \qquad \therefore \qquad PR = 8$

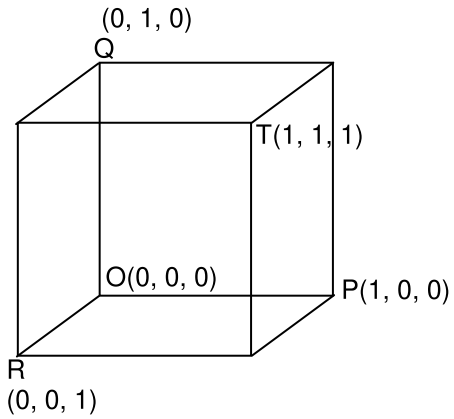
$R(3, 3, -4)$

45. Consider the cube in the first octant with sides OP, OQ and OR of length 1, along the x-axis, y-axis and z-axis, respectively, where O(0, 0, 0) is the origin. Let $S\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ be the centre of the cube and T be the vertex of the cube opposite to the origin O such that S lies on the diagonal OT. If $\vec{p} = \overrightarrow{SP}$, $\vec{q} = \overrightarrow{SQ}$, $\vec{r} = \overrightarrow{SR}$ and $\vec{t} = \overrightarrow{ST}$, then the value of $|(\vec{p}\times\vec{q})\times(\vec{r}\times\vec{t})|$ is ______ . **[JEE(Advanced) 2018, Paper-2,(3, 0), 60]**

प्रथम अष्ठांश (first octant) में एक ऐसे घन (cube) पर विचार कीजिये, जिसकी भुजाओं (sides) OP, OQ और OR की लम्बाई 1 है और जो क्रमशः x-अक्ष (axis), y-अक्ष और z-अक्ष के अनुदिश (along) हैं, जहॉ O(0, 0, 0) मूलबिन्दु (origin) है। माना कि घन का केंद्र (centre) $S\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ है, और शीर्ष (vertex) T मूलबिन्दु O के सम्मुख (opposite) वाला वह शीर्ष है कि बिंदु S विकर्ण (diagonal) OT पर स्थित है। यदि $\vec{p} = \overrightarrow{SP}$, $\vec{q} = \overrightarrow{SQ}$, $\vec{r} = \overrightarrow{SR}$ और $\vec{t} = \overrightarrow{ST}$ तब $|(\vec{p}\times\vec{q})\times(\vec{r}\times\vec{t})|$ का मान है ______ ।

Ans. **(0.5)**

Sol.



point बिन्दु $S\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$

point बिन्दु T(1, 1, 1)

$$\vec{p} = \overrightarrow{SP} = \frac{\hat{i}-\hat{j}-\hat{k}}{2}$$

$$\vec{q} = \overrightarrow{SQ} = \frac{-\hat{i}+\hat{j}-\hat{k}}{2}$$

$$\vec{r} = \overrightarrow{SR} = \frac{-\hat{i}-\hat{j}+\hat{k}}{2}$$

$$\vec{t} = \overrightarrow{ST} = \frac{\hat{i}+\hat{j}+\hat{k}}{2}$$

Now अब $\vec{p}\times\vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & -1 \\ -1 & 1 & -1 \end{vmatrix} \times \frac{1}{4} = \frac{1}{4}(2\hat{i}+2\hat{j}) = \frac{\hat{i}+\hat{j}}{2}$

$$\vec{r}\times\vec{t} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} \times \frac{1}{4} = \frac{-2\hat{i}+2\hat{j}}{4} = \frac{-\hat{i}+\hat{j}}{2}$$

Now अब $(\vec{p}\times\vec{q})\times(\vec{r}\times\vec{t}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{vmatrix} \times \frac{1}{4} = \frac{\hat{k}}{2} \Rightarrow \left|(\vec{p}\times\vec{q})\times(\vec{r}\times\vec{t})\right| = \frac{1}{2} = 0.5$

46. Let L_1 and L_2 denote the lines $\vec{r} = \hat{i} + \lambda(-\hat{i} + 2\hat{j} + 2\hat{k}), \lambda \in R$ and $\vec{r} = \mu(2\hat{i} - \hat{j} + 2\hat{k}), \mu \in R$ respectively. If L_3 is a line which is perpendicular to both L_1 and L_2 and cuts both of them, then which of the following options describe(s) L_3 ? **[Vector]** **[T] {Vector[VT-SL]-T-305}**

[JEE(Advanced) 2019, Paper-1,(4, –1)/62]

माना कि L_1 और L_2 क्रमशः निम्न रेखाएं है : $\vec{r} = \hat{i} + \lambda(-\hat{i} + 2\hat{j} + 2\hat{k}), \lambda \in R$ और $\vec{r} = \mu(2\hat{i} - \hat{j} + 2\hat{k}), \mu \in R$ यदि L_3 एक रेखा है जो L_1 और L_2 दोनों के लम्बवत है और दोनों को काटती है, तब निम्नलिखित विकलपों में से कौन सा (से) L_3 को निरूपित करता (करते) है (हैं) ?

(A) $\vec{r} = \frac{1}{3}(2\hat{i} + \hat{k}) + t(2\hat{i} + 2\hat{j} - \hat{k}), t \in R$

(B) $\vec{r} = \frac{2}{9}(4\hat{i} + \hat{j} + \hat{k}) + t(2\hat{i} + 2\hat{j} - \hat{k}), t \in R$

(C) $\vec{r} = \frac{2}{9}(2\hat{i} - \hat{j} + 2\hat{k}) + t(2\hat{i} + 2\hat{j} - \hat{k}), t \in R$

(D) $\vec{r} = t(2\hat{i} + 2\hat{j} - \hat{k}), t \in R$

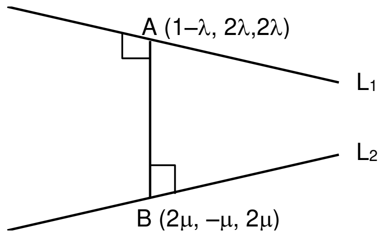
Ans. **(ABC)**

Sol. Both given lines are skew lines.

So direction ratios of any line perpendicular to these lines are $6\hat{i}+6\hat{j}-3\hat{k}$

<2, 2, –1>

Points at shortest distance between given lines are



$\overrightarrow{AB} \perp$ line L_1

$\overrightarrow{AB} \perp$ line L_2

So $A\left(\frac{8}{9},\frac{2}{9},\frac{2}{9}\right)$

Now equation of required line $\vec{r} = \left(\frac{8}{9}\hat{i}+\frac{2}{9}\hat{j}+\frac{2}{9}\hat{k}\right)+\alpha\left(2\hat{i}+2\hat{j}-\hat{k}\right)$

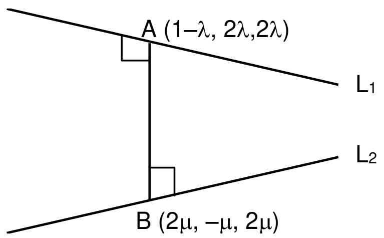
Now by option B, C, D are correct.

Hindi. दी गई दोनों रेखाएं विषम तलीय रेखाएं है।

इसलिए किसी रेखा द्विकअनुपात इन रेखाओं के लम्बवत है। $6\hat{i}+6\hat{j}-3\hat{k}$

$<2, 2, -1>$

दी गई रेखाओं के मध्य लघुतम दूरी



$\overrightarrow{AB} \perp$ रेखा L_1

$\overrightarrow{AB} \perp$ रेखा L_2

इसलिए $A\left(\frac{8}{9},\frac{2}{9},\frac{2}{9}\right)$

$\vec{r} = \left(\frac{8}{9}\hat{i}+\frac{2}{9}\hat{j}+\frac{2}{9}\hat{k}\right)+\alpha\left(2\hat{i}+2\hat{j}-\hat{k}\right)$

47. Three lines are given by **{Vector[VT-PL]-M-303}**

$\vec{r} = \lambda\hat{i},\ \lambda \in R$

$\vec{r} = \mu(\hat{i}+\hat{j})$, n $\in$ R and $\vec{r} = \nu(\hat{i}+\hat{j}+\hat{k}),\ \nu \in R$

Let the lines cut the plane x + y + z = 1 at he points A, B and C respectively. If the area of the triangle ABC is Δ then the value of $(6\Delta)^2$ equals .... **[JEE(Advanced) 2019, Paper-1,(4, –1)/62]**

तीन रेखाएं क्रमशः

$\vec{r} = \lambda\hat{i},\ \lambda \in R$

$\vec{r} = \mu(\hat{i}+\hat{j})$, n $\in$ R and $\vec{r} = \nu(\hat{i}+\hat{j}+\hat{k}),\ \nu \in R$

द्वारा दी गयी हैं। माना कि रेखाएं समतल (plane) x + y + z = 1 को क्रमशः बिन्दुओं A, B और C पर काटती हैं। यदि त्रिभुज ABC का क्षेत्रफल Δ है तब $(6\Delta)^2$ का मान बराबर ....

Ans. **(0.75)**

Sol. Put $(\lambda, 0, 0)$ in रखने पर x + y + z = 1 में $\Rightarrow \lambda = 1 \Rightarrow P(1, 0, 0)$

Put $(\mu, \mu, 0)$ रखने पर $\Rightarrow 2\mu = 1$ $\Rightarrow Q\left(\frac{1}{2},\frac{1}{2},0\right)$

Put (γ,γ,γ) रखने पर $\Rightarrow \gamma = \frac{1}{3}$ $\Rightarrow R\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$

Area of triangle त्रिभुज PQR का क्षेत्रफल $= \frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = \frac{1}{2}\left|\left(\frac{\hat{i}-\hat{j}}{2}\right)\times\left(\frac{2\hat{i}-\hat{j}-\hat{k}}{3}\right)\right| =$

$\frac{1}{12}\left|\hat{i}+\hat{j}+\hat{k}\right| = \frac{\sqrt{3}}{12} \Rightarrow\ (6\Delta)^2 = 0.75$

48. Three lines

$L_1 : \vec{r} = \lambda\hat{i},\ \lambda \in R$

$L_2 : \vec{r} = \hat{k}+\mu\hat{j},\ \mu \in R$ and

$L_3 : \vec{r} = \hat{i}+\hat{j}+\nu\hat{k},\ \nu \in R$.

are given. For which point(s) Q on L_2 can we find a point P on L_1 and a point R on L_3 so that P,Q and R are collinear. **[Vector_M] [JEE(Advanced) 2019, Paper-2 ,(4, –1)/62]{[VT-CL]-M-303}**

तीन रेखाएं

$L_1 : \vec{r} = \lambda\hat{i}$, $\lambda \in R$

$L_2 : \vec{r} = \hat{k} + \mu\hat{j}$, $\mu \in R$ तथा

$L_3 : \vec{r} = \hat{i} + \hat{j} + \nu\hat{k}$, $\nu \in R$.

दी गयीं है। L_2 के किस बिन्दु (किन बिन्दुओ) Q के लिए हम L_1 पर एक बिन्दु P, और L_3 पर एक बिन्दु R प्राप्त कर सकते हैं। ताकि P, Q और R सरेख (collinear) हो जाएँ ?

(A) $\hat{k} + \frac{1}{2}\hat{j}$ (B) $\hat{k} + \hat{j}$ (C) $\hat{k}$ (D) $\hat{k} - \frac{1}{2}\hat{j}$

Ans. **(AD)**

Sol. $P(\lambda, 0, 0)$

$Q(0, \mu, 1)$

$R(1, 1, \gamma)$

$\overrightarrow{PQ} = k\overrightarrow{PR} \quad \Rightarrow \frac{\lambda}{\lambda - 1} = \frac{-\mu}{-1} = \frac{-1}{-\gamma}$

$\Rightarrow 1 + \frac{1}{\lambda - 1} = \mu = \frac{1}{\gamma} \quad \Rightarrow \mu$ cannot take value of 1 and 0

$\Rightarrow \mu$, 1 और 0 मान नहीं ले सकता है।

49. Let $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$ be two vectors. Consider a vector $\vec{c} = \alpha\vec{a} + \beta\vec{b}$, $\alpha, \beta \in R$. If the projection of $\vec{c}$ on the vector $(\vec{a} + \vec{b})$ is $3\sqrt{2}$, then the minimum value of $(\vec{c} - (\vec{a} \times \vec{b})).\vec{c}$ equal to

यदि $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$ तथा $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ दो सदिश दिये हुए है माना एक सदिश $\vec{c}$ इस प्रकार है कि $\vec{c} = \alpha\vec{a} + \beta\vec{b}$, $\alpha, \beta \in R$. यदि सदिश $\vec{c}$ का सदिश $(\vec{a} + \vec{b})$ पर प्रक्षेप $3\sqrt{2}$ है तो $(\vec{c} - (\vec{a} \times \vec{b})).\vec{c}$ का न्यूनतम मान होगा–

Ans. **(18.00)** **[JEE(Advanced) 2019, Paper-2 ,(4, –1)/62]**
[Vector_M] **{[VT-DP]-M-303}**

Sol. $\vec{c} = \alpha(2\hat{i} + \hat{j} - \hat{k}) + \beta(\hat{i} + 2\hat{j} + \hat{k})$

$\Rightarrow \quad \vec{c} = (2\alpha + \beta)\hat{i} + (\alpha + 2\beta)\hat{j} + (\beta - \alpha)\hat{k}$

$\frac{\vec{c}.(\vec{a} + \vec{b})}{|\vec{a} + \vec{b}|} = 3\sqrt{2} \Rightarrow 9(\alpha + \beta) = 18 \Rightarrow \alpha + \beta = 2$

$(\vec{c} - \vec{a} \times \vec{b}).\vec{c} = (\alpha\vec{a} + \beta\vec{b} - \vec{a} \times \vec{b}) . (\alpha\vec{a} + \beta\vec{b}) = 6\alpha^2 + 6\alpha\beta + 6\beta^2 = 6[\alpha^2 + \alpha(2 - \alpha) + (2 - \alpha)^2]$

$= 6(\alpha^2 - 2\alpha + 4) =$ min value 18

PART - II : JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

भाग - II : JEE (MAIN) / AIEEE (पिछले वर्षों) के प्रश्न

1. Let $\vec{a} = \hat{j} - \hat{k}$ and $\vec{c} = \hat{i} - \hat{j} - \hat{k}$. Then the vector $\vec{b}$ satisfying $\vec{a} \times \vec{b} + \vec{c} = \vec{0}$ and $\vec{a}.\vec{b} = 3$ is

माना $\vec{a} = \hat{j} - \hat{k}$ तथा $\vec{c} = \hat{i} - \hat{j} - \hat{k}$ तो सदिश $\vec{b}$ जो $\vec{a} \times \vec{b} + \vec{c} = \vec{0}$ तथा $\vec{a}.\vec{b} = 3$ को संतुष्ट करता है, होगा–

[AIEEE 2010 (4, –1), 144]

(1) $2\hat{i} - \hat{j} + 2\hat{k}$ (2) $\hat{i} - \hat{j} - 2\hat{k}$ (3) $\hat{i} + \hat{j} - 2\hat{k}$ (4*) $-\hat{i} + \hat{j} - 2\hat{k}$

Ans. **(4)**

Sol. Since $\vec{a} \times \vec{b} + \vec{c} = \vec{0}$

$\Rightarrow \vec{a} \times (\vec{a} \times \vec{b}) + \vec{a} \times \vec{c} = \vec{0} \Rightarrow (\vec{a} \,.\, \vec{b})\vec{a} - (\vec{a} \,.\, \vec{a})\,\vec{b} + \vec{a} \times \vec{c} = \vec{0}$

Since $\vec{a} \times \vec{c} = -2\hat{i} - \hat{j} - \hat{k}$

$\Rightarrow 3(\hat{j} - \hat{k}) - 2\vec{b} - 2\hat{i} - \hat{j} - \hat{k} = \vec{0} \Rightarrow \vec{b} = -\hat{i} + \hat{j} - 2\hat{k}$

Hence correct option is (4)

अतः सही विकल्प (4) है।

2. If the vectors $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$, $\vec{b} = 2\hat{i} + 4\hat{j} + \hat{k}$ and $\vec{c} = \lambda\hat{i} + \hat{j} + \mu\hat{k}$ are mutually orthogonal, then (λ, μ) =

यदि सदिश $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$, $\vec{b} = 2\hat{i} + 4\hat{j} + \hat{k}$ तथा $\vec{c} = \lambda\hat{i} + \hat{j} + \mu\hat{k}$ परस्पर लम्बकोणीय है, तो (λ, μ) बराबर है–

[AIEEE 2010 (4, –1), 144]

(1) (2, – 3) (2) (–2, 3) (3) (3, – 2) (4*) (–3, 2)

Ans. **(4)**

Sol. $\because$ $\vec{a}, \vec{b}, \vec{c}$ are mutually orthogonal परस्पर लम्बवत् है।

$\therefore$ $\vec{a} \,.\, \vec{c} = 0 \Rightarrow \lambda - 1 + 2\mu = 0$ ....(i)

and तथा

$\vec{b} \,.\, \vec{c} = 0 \Rightarrow 2\lambda + 4 + \mu = 0$ ....(ii)

solving (i) and (ii), we get $\lambda = -3$ and $\mu = 2$

(i) व (ii) को हल करने पर $\lambda = -3$ तथा $\mu = 2$

Hence correct option is (4)

अतः सही विकल्प (4) है।

3. **Statement -1 :** The point A(3, 1, 6) is the mirror image of the point B(1, 3, 4) in the plane x – y + z = 5.

Statement -2 : The plane x – y + z = 5 bisects the line segment joining A(3, 1,6) and B(1, 3, 4).

(1*) Statement -1 is true, Statement-2 is true ; Statement -2 is not a correct explanation for Statement -1.

(2) Statement-1 is true, Statement-2 is false. **[AIEEE 2009 (4, –1), 144]**

(3) Statement -1 is false, Statement -2 is true.

(4) Statement -1 is true, Statement -2 is true; Statement-2 is a correct explanation for Statement-1.

प्रकथन-1 : समतल x – y + z = 5 में बिन्दु B(1, 3, 4) का दर्पण प्रतिबिम्ब बिन्दु A(3, 1, 6) है।

प्रकथन-2 : समतल x – y + z = 5, A(3, 1,6) तथा B(1, 3, 4) को मिलाने वाले रेखाखण्ड को समद्विभाजित करता है।

(1*) प्रकथन-1 सत्य है, प्रकथन-2 सत्य है ; प्रकथन-2, प्रकथन-1 की सही व्याख्या नहीं है।

(2) प्रकथन-1 सत्य है, प्रकथन-2 मिथ्या है।

(3) प्रकथन-1 मिथ्या है, प्रकथन-2 सत्य है।

(4) प्रकथन-1 सत्य है, प्रकथन-2 सत्य है ; प्रकथन-2, प्रकथन-1 की सही व्याख्या है।

Ans. **(1)**

Sol. Let image be (α, β, γ)

$$\frac{\alpha-1}{1} = \frac{\beta-3}{-1} = \frac{\gamma-4}{1} = -2\left(\frac{1-3+4-5}{3}\right)$$

$\Rightarrow \quad \frac{\alpha-1}{1} = \frac{\beta-3}{-1} = \frac{\gamma-4}{1} = 2$

$\Rightarrow \quad \alpha = 3, \beta = 1, \gamma = 6$

$\Rightarrow$ A(3, 1, 6) statement 1 is true

Now midpoint of A(3, 1, 6) and B(1, 3, 4) is (2, 2, 5)

equation of plane is $x - y + z = 5$

coordinates of midpoint lies on the plane so plane bisects the line segment AB. But it is not correct explanation of statement-1

Hence correct option is (1)

Hindi माना प्रतिबिम्ब (α, β, γ) है तो

$$\frac{\alpha-1}{1} = \frac{\beta-3}{-1} = \frac{\gamma-4}{1} = -2\left(\frac{1-3+4-5}{3}\right)$$

$\Rightarrow \quad \frac{\alpha-1}{1} = \frac{\beta-3}{-1} = \frac{\gamma-4}{1} = 2$

$\Rightarrow \quad \alpha = 3, \beta = 1, \gamma = 6$

$\Rightarrow$ A(3, 1, 6) कथन -1 सत्य है।

अब A(3, 1, 6) और B(1, 3, 4) का मध्य बिन्दु (2, 2, 5) है।

समतल का समीकरण $x - y + z = 5$

मध्य बिन्दु के निर्देशांक समतल पर स्थित है इसलिए समतल रेखाखण्ड AB को समद्विभाजित करता है परन्तु यह कथन -1 का सही स्पष्टीकरण नहीं है।

अतः सही विकल्प (1) है।

4. A line AB in three-dimensional space makes angles 45º and 120º with the positive x-axis and the positive
y-axis respectively. If AB makes an acute angle θ with the positive z-axis, then θ equal

त्रिविम समष्टि में रेखा AB, x-अक्ष की धनात्मक–दिशा के साथ 45º का कोण तथा y-अक्ष की धनात्मक–दिशा के साथ 120º का कोण बनाती है। यदि AB, z-अक्ष की धनात्मक दिशा से न्यून कोण θ बनाती है, तो θ बराबर है–

[AIEEE 2010 (4, –1), 144]

(1) 45º (2*) 60º (3) 75º (4) 30º

Ans. **(2)**

Sol. $\ell = \frac{1}{\sqrt{2}}, \quad m = -\frac{1}{2}$

$\ell^2 + m^2 + n^2 = 1 \Rightarrow \quad n^2 = \frac{1}{4} \Rightarrow \quad n = \pm\frac{1}{2}$

$\cos\theta = \frac{1}{2}, \quad \theta = 60º$ Hence correct option is (2) अतः सही विकल्प (2) है।

5. If $\vec{a} = \frac{1}{\sqrt{10}}(3\hat{i}+\hat{k})$ and $\vec{b} = \frac{1}{7}(2\hat{i}+3\hat{j}-6\hat{k})$, then the value of $(2\vec{a}-\vec{b}).[(\vec{a}\times\vec{b})\times(\vec{a}+2\vec{b})]$ is:

यदि $\vec{b} = \frac{1}{7}(2\hat{i}+3\hat{j}-6\hat{k})$ तथा $\vec{b} = \frac{1}{7}(2\hat{i}+3\hat{j}-6\hat{k})$, तो $(2\vec{a}-\vec{b}).[(\vec{a}\times\vec{b})\times(\vec{a}+2\vec{b})]$ बराबर है :

[AIEEE 2011, I, (4, –1), 120]

(1*) – 5 (2) –3 (3) 5 (4) 3

Sol. **(1)**

$(2\vec{a} - \vec{b}).[(\vec{a} \times \vec{b}) \times (\vec{a} + 2\vec{b})]$

$= -(2\vec{a} - \vec{b}).[(\vec{a} + 2\vec{b}) \times (\vec{a} \times \vec{b})]$

$= -(2\vec{a} - \vec{b}).[(\vec{a} + 2\vec{b}).\vec{b})\vec{a} - ((\vec{a} + 2\vec{b}).\vec{a})\vec{b}]$
$= -(2\vec{a} - \vec{b}).[(\vec{a}.\vec{b}) + 2.\vec{b})\vec{a} - (\vec{a}.\vec{a} + 2\vec{b}.\vec{a})\vec{b})]$
$= -(2\vec{a} - \vec{b}).[0 + 2\vec{a} - (0 + \vec{b})]$
$= -(2\vec{a} - \vec{b}).(2\vec{a} - \vec{b})$
$= -(2\vec{a} - \vec{b})^2 = -4\vec{a}^2 + 4\vec{a}.\vec{b} - \vec{b}^2$
$= -4 + 0 - 1 = -5$ **Ans.**

6. The vectors $\vec{a}$ and $\vec{b}$ are not perpendicular and $\vec{c}$ and $\vec{d}$ are two vectors satisfying : $\vec{b}\times\vec{c} = \vec{b}\times\vec{d}$ and $\vec{a}.\vec{d} = 0$. Then the vector $\vec{d}$ is equal to : **[AIEEE 2011, I, (4, –1), 120]**

$\vec{a}$ तथा $\vec{b}$ लम्बवत नहीं है तथा $\vec{c}$ तथा $\vec{d}$ इस प्रकार के है कि $\vec{b}\times\vec{c} = \vec{b}\times\vec{d}$ तथा $\vec{a}\,.\,\vec{d} = 0$ है, तो $\vec{d}$ बराबर है:

[AIEEE 2011, I, (4, –1), 120]

(1) $\vec{b} - \left(\frac{\vec{b}.\vec{c}}{\vec{a}.\vec{d}}\right)\vec{c}$ (2) $\vec{c} + \left(\frac{\vec{a}.\vec{c}}{\vec{a}.\vec{b}}\right)\vec{b}$ (3) $\vec{b} + \left(\frac{\vec{b}.\vec{c}}{\vec{a}.\vec{b}}\right)\vec{c}$ (4*) $\vec{c} - \left(\frac{\vec{a}.\vec{c}}{\vec{a}.\vec{b}}\right)\vec{b}$

Sol. **(4)**
$\vec{a}\,.\,\vec{b} \neq 0$, $\vec{b}\times\vec{c} = \vec{b}\times\vec{d}$, $\vec{a}\,.\,\vec{d} = 0$
$(\vec{b}\times\vec{c}) \times \vec{a} = (\vec{b}\times\vec{d}) \times \vec{a}$
$(\vec{b}.\vec{a}) - (\vec{c}.\vec{a}) = (\vec{b}\,.\,\vec{a})\,\vec{d} - (\vec{d}\,.\,\vec{a})\,\vec{b}$
$\vec{d} = \vec{c} - \left(\frac{\vec{a}\,.\,\vec{c}}{\vec{a}.\,\vec{b}}\right)\vec{b}$

7. If the vector $p\hat{i} + \hat{j} + \hat{k}$, $\hat{i} + q\hat{j} + \hat{k}$ and $\hat{i} + \hat{j} + r\hat{k}$ $(p \neq q \neq r \neq 1)$ are coplanar, then the value of pqr – (p+q+r) is-

यदि सदिश $p\hat{i} + \hat{j} + \hat{k}$, $\hat{i} + q\hat{j} + \hat{k}$ तथा $\hat{i} + \hat{j} + r\hat{k}$ $(p \neq q \neq r \neq 1)$ समतलीय है, तो pqr – (p+q+r) का मान है :

[AIEEE 2011, II, (4, –1), 120]

(1) 2 (2) 0 (3) –1 (4*) –2

Sol. **(4)**
$$\begin{vmatrix} p & 1 & 1 \\ 1 & q & 1 \\ 1 & 1 & r \end{vmatrix} = 0$$
$\Rightarrow$ p (qr – 1) + 1(1 – r) + 1(1 – q) = 0
$\Rightarrow$ pqr – p + 1– r + 1 – q = 0
$\Rightarrow$ pqr – (p + q + r) = – 2

8✎. Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three non-zero vectors which are pairwise non-collinear. If $\vec{a} + 3\vec{b}$ is collinear with $\vec{c}$ and $\vec{b} + 2\vec{c}$ is collinear with $\vec{a}$, then $\vec{a} + 3\vec{b} + 6\vec{c}$ is :
माना $\vec{a}$, $\vec{b}$, $\vec{c}$ तीन शून्येतर सदिश है जो युग्मतः संरेख नहीं है। यदि $\vec{a} + 3\vec{b}$ एवं $\vec{c}$ संरेखीय है तथा $\vec{b} + 2\vec{c}$ एवं $\vec{a}$ संरेखीय हैं, तो $\vec{a} + 3\vec{b} + 6\vec{c}$ बराबर है : **[AIEEE 2011, II, (4, –1), 120]**

(1) $\vec{a}$ (2) $\vec{c}$ (3*) $\vec{0}$ (4) $\vec{a} + \vec{c}$

Sol. **(3)**
$\vec{a} + 3\vec{b} = \lambda\vec{c}$ ....(1)
$\vec{b} + 2\vec{c} = \mu\vec{a}$ ....(2)
(1) – 3(2) gives $(1 + 3\mu)\,\vec{a} - (\lambda + 6)\,\vec{c} = 0$

As $\vec{a}$ and $\vec{c}$ are non collinear

$\vec{a}$ तथा $\vec{c}$ असंरेखीय है। अतः

$\therefore$ 1 + 3μ = 0 and और λ + 6 = 0

From (1) से $\vec{a}+3\vec{b}+6\vec{c}=\vec{0}$

9. If the angle between the line x = $\frac{y-1}{2}$ = $\frac{z-3}{\lambda}$ and the plane x + 2y + 3z = 4 is $\cos^{-1}\left(\sqrt{\frac{5}{14}}\right)$, then λ equals:

यदि रेखा x = $\frac{y-1}{2}$ = $\frac{z-3}{\lambda}$ तथा समतल x + 2y + 3z = 4 के बीच का कोण $\cos^{-1}\left(\sqrt{\frac{5}{14}}\right)$ है, तो λ का मान है:

[AIEEE 2011, I, (4, –1), 120]

(1*) $\frac{2}{3}$ (2) $\frac{3}{2}$ (3) $\frac{2}{5}$ (4) $\frac{5}{3}$

Sol. **(1)**

$\frac{x-0}{1}=\frac{y-1}{2}=\frac{z-3}{\lambda}$ .... (1)

x + 2y + 3z = 4 .... (2)

Angle between the line and plane is

$$\cos(90° - \theta) = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

$\Rightarrow$ $\sin\theta = \frac{1+4+3\lambda}{\sqrt{14}\times\sqrt{5+\lambda^2}} = \frac{5+3\lambda}{\sqrt{14}\times\sqrt{5+\lambda^2}}$ .... (3)

But given that angle between line and plane is

$\theta = \cos^{-1}\left(\sqrt{\frac{5}{14}}\right) = \sin^{-1}\left(\frac{3}{\sqrt{14}}\right)$

$\Rightarrow$ $\sin\theta = \frac{3}{\sqrt{14}}$

$\therefore$ from (3)

$\frac{3}{\sqrt{14}} = \frac{5+3\lambda}{\sqrt{14}\times\sqrt{5+\lambda^2}}$

$\Rightarrow$ 9(5 + λ²) = 25 + 9λ² + 30 λ

$\Rightarrow$ 30λ = 20

λ = $\frac{2}{3}$ **Ans.**

Hindi $\frac{x-0}{1}=\frac{y-1}{2}=\frac{z-3}{\lambda}$ .... (1)

x + 2y + 3z = 4 .... (2)

रेखा तथा समतल के बीच का कोण

$\cos(90° - \theta) = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$ $\Rightarrow$ $\sin\theta = \frac{1+4+3\lambda}{\sqrt{14}\times\sqrt{5+\lambda^2}} = \frac{5+3\lambda}{\sqrt{14}\times\sqrt{5+\lambda^2}}$ .... (3)

परन्तु दिया है, कि रेखा तथा समतल के बीच का कोण

$\theta = \cos^{-1}\left(\sqrt{\frac{5}{14}}\right) = \sin^{-1}\left(\frac{3}{\sqrt{14}}\right)$

$\Rightarrow$ $\sin\theta = \frac{3}{\sqrt{14}}$

$\therefore$ from (3)

$$\frac{3}{\sqrt{14}} = \frac{5+3\lambda}{\sqrt{14}\times\sqrt{5+\lambda^2}}$$

$\Rightarrow$ $9(5 + \lambda^2) = 25 + 9\lambda^2 + 30\,\lambda$

$\Rightarrow$ $30\lambda = 20$

$\lambda = \frac{2}{3}$ **Ans.**

10. **Statement-1 :** The point A(1, 0, 7) is the mirror image of the point B(1, 6, 3) in the line : $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$

[AIEEE 2011, I, (4, –1), 120]

Statement-2 : The line : $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ bisects the line segment joining A(1, 0, 7) and B(1, 6, 3).

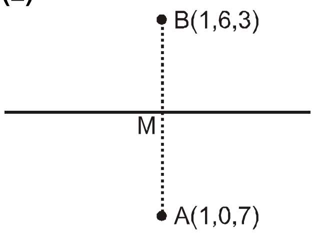
(1) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.
(2*) Statement-1 is true, Statement-2 is true; Statement-2 is **not** a correct explanation for Statement-1.
(3) Statement-1 is true, Statement-2 is false.
(4) Statement-1 is false, Statement-2 is true.

कथन-1 : $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ रेखा में बिन्दु B(1, 6, 3) का दर्पण प्रतिबिंब (mirror image) बिन्दु A(1, 0, 7) है।

कथन-2 : $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ रेखा बिन्दुओं A(1, 0, 7) तथा B(1, 6, 3) को मिलाने वाले रेखाखंड का समद्विभाजन करती है। **[AIEEE 2011, I, (4, –1), 120]**

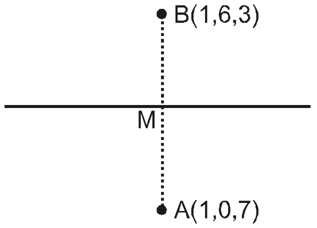
(1) कथन-1 सत्य है, कथन-2 सत्य है। कथन-2, कथन-1 की सही व्याख्या है।
(2*) कथन-1 सत्य है, कथन-2 सत्य है। कथन-2, कथन-1 की सही व्याख्या **नहीं** है।
(3) कथन-1 सत्य है, कथन-2 असत्य है।
(4) कथन-1 असत्य है, कथन-2 सत्य है।

Sol. **(2)**



Mid- point of AB $\equiv$ M(1,3,5)
M lies on line
Direction ratios of AB is < 0, 6, – 4 >
Direction ratios of given line is < 1, 2, 3 >
As AB is perpendicular to line
$\therefore$ 0.1 + 6.2 – 4.3 = 0

Hindi



AB का मध्य बिन्दु $\equiv$ M(1,3,5)
M रेखा पर स्थित है।
AB के दिक् अनुपात < 0, 6, – 4 >

दी गई रेखा के दिक् अनुपात < 1, 2, 3 >
चूंकि दी गई रेखा AB के लम्बवत् है।
$\therefore$ 0.1 + 6.2 – 4.3 = 0

11. The distance of the point (1, –5, 9) from the plane x – y + z = 5 measured along a straight line x = y = z is :
बिन्दु (1, –5, 9) की समतल x – y + z = 5 से दूरी जो सरल रेखा x = y = z के अनुदिश मापी गयी है, है :
[AIEEE 2011, II, (4, –1), 120]

(1*) $10\sqrt{3}$ (2) $5\sqrt{3}$ (3) $3\sqrt{10}$ (4) $3\sqrt{5}$

Sol. **(1)**
Line through P(1, – 5, 9) parallel to x = y = z is $\frac{x-1}{1} = \frac{y+5}{1} = \frac{z-9}{1} = \lambda$ (say)
बिन्दु P(1, – 5, 9) से गुजरने वाली तथा x = y = z के समान्तर रेखा $\frac{x-1}{1} = \frac{y+5}{1} = \frac{z-9}{1} = \lambda$ (माना) है।
Q (x = 1 + λ, y = – 5 + λ, z = 9 + λ)
Given planeदिया गया समतल x – y + z = 5
$\therefore$ 1 + λ + 5 – λ + 9 + λ = 5
$\Rightarrow$ λ = –10
$\therefore$ Q(–9, – 15, – 1)
$\therefore$ PQ = $\sqrt{300}$
= $\sqrt{300}$ = $10\sqrt{3}$

12. The length of the perpendicular drawn from the point (3, –1, 11) to the line $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ is :
बिन्दु (3, –1, 11) से रेखा $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ पर डाले गए लम्ब की लम्बाई है : **[AIEEE 2011, II, (4, –1), 120]**

(1) $\sqrt{29}$ (2) $\sqrt{33}$ (3*) $\sqrt{53}$ (4) $\sqrt{66}$

Sol. **(3)**
Let feet of perpendicular is (2α , 3α + 2, 4α + 3)
माना लम्बपाद के निर्देशांक (2α , 3α + 2, 4α + 3) है।
$\Rightarrow$ D′ ratio of the perpendicular line < 2α – 3, 3α + 3, 4α – 8 >
लम्बवत् रेखा के दिक् अनुपात < 2α – 3, 3α + 3, 4α – 8 >
and D′ ratio of the line < 2, 3, 4 >
और दी गई रेखा के दिक् अनुपात < 2, 3, 4 >
$\Rightarrow$ 2(2α – 3) + 3 (3α + 3) + 4 (4α – 8) = 0
$\Rightarrow$ 29 α – 29 = 0
$\Rightarrow$ α = 1
$\Rightarrow$ feet of perpendicular is (2, 5, 7)
लम्बपाद के निर्देशांक (2, 5, 7)
$\Rightarrow$ length perpendicular is लम्ब की लम्बाई $\sqrt{1^2+6^2+4^2} = \sqrt{53}$

13. Let $\hat{a}$ and $\hat{b}$ be two unit vectors. If the vectors $\vec{c} = \hat{a} + 2\hat{b}$ and $\vec{d} = 5\hat{a} - 4\hat{b}$ are perpendicular to each other, then the angle between $\hat{a}$ and $\hat{b}$ is :
माना $\hat{a}$ तथा $\hat{b}$ दो मात्रक सदिश हैं। यदि सदिश $\vec{c} = \hat{a} + 2\hat{b}$ और $\vec{d} = 5\hat{a} - 4\hat{b}$ परस्पर लंबवत् हैं, तो $\hat{a}$ तथा $\hat{b}$ के बीच का कोण है : **[AIEEE-2012, (4, –1)/120]**

(1) $\frac{\pi}{6}$ (2) $\frac{\pi}{2}$ (3*) $\frac{\pi}{3}$ (4) $\frac{\pi}{4}$

Sol. **Ans. (3)**
$\vec{c} = \hat{a} + 2\hat{b}$

$\vec{d} = 5\hat{a} - 4\hat{b}$

$\vec{c}.\vec{d} = 0$

$\Rightarrow \quad (\hat{a} + 2\hat{b}).(5\hat{a} - 4\hat{b}) = 0 \qquad \Rightarrow \quad 5 + 6\hat{a}.\hat{b} - 8 = 0$

$\Rightarrow \quad \hat{a}.\hat{b} = \frac{1}{2} \qquad \Rightarrow \quad \theta = \frac{\pi}{3}$

14. A equation of a plane parallel to the plane $x - 2y + 2z - 5 = 0$ and at a unit distance from the origin is :

समतल $x - 2y + 2z - 5 = 0$ के समांतर तथा मूल बिंदु से मात्रक दूरी पर एक समतल का समीकरण है–

[AIEEE 2012, (4, –1), 120]

(1*) $x - 2y + 2z - 3 = 0$ (2) $x - 2y + 2z + 1 = 0$ (3) $x - 2y + 2z - 1 = 0$ (4) $x - 2y + 2z + 5 = 0$

Sol. Ans. (1)

Equation of parallel plane $x - 2y + 2z + d = 0$

Now $\left|\frac{d}{\sqrt{1^2 + 2^2 + 2^2}}\right| = 1$

$d = \pm 3$

So equation required plane $x - 2y + 2z \pm 3 = 0$

Hindi Ans. (1)

समान्तर समतल का समीकरण $x - 2y + 2z + d = 0$

अब $\left|\frac{d}{\sqrt{1^2 + 2^2 + 2^2}}\right| = 1$

$d = \pm 3$

अतः अभीष्ट समतल का समीकरण है $x - 2y + 2z \pm 3 = 0$

15. If the line $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ intersect, then k is equal to :

यदि रेखाएँ $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ तथा $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ परस्पर प्रतिच्छेद करती हैं, तो k बराबर है : :

[AIEEE 2012, (4, –1), 120]

(1) – 1 (2) $\frac{2}{9}$ (3*) $\frac{9}{2}$ (4) 0

Sol. Ans. (3)

$\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$

$\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$

$\vec{a}\,(1, -1, 1)\,; \quad \vec{r} = \vec{a} + \lambda\vec{b}$

$\vec{b}\,(2, 3, 4)$

$\vec{c}\,(3, k, 0); \quad \vec{r} = \vec{c} + \mu\vec{d}$

$\vec{d}\,(1, 2, 1)$

These lines will intersect if lines are coplaner

$\vec{a} - \vec{c}\,,\ \hat{b}\ \&\ \vec{d}$ are coplaner

$\therefore \quad [\vec{a} - \vec{c},\ \vec{b},\ \vec{d}] = 0$

$$\begin{vmatrix} 2 & k+1 & -1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad 2(-5) - (k + 1)(-2) - 1(1) = 0$

$\Rightarrow \quad 2(k + 1) = 11$

$\Rightarrow \quad k = \dfrac{9}{2}$

Hindi $\dfrac{x-1}{2} = \dfrac{y+1}{3} = \dfrac{z-1}{4}$

$\dfrac{x-3}{1} = \dfrac{y-k}{2} = \dfrac{z}{1}$

$\vec{a}\,(1, -1, 1)$; $\quad \vec{r} = \vec{a} + \lambda\vec{b}$

$\vec{b}\ (2, 3, 4)$

$\vec{c}\ (3, k, 0)$; $\quad \vec{r} = \vec{c} + \mu\vec{d}$

$\vec{d}\ (1, 2, 1)$

सरल रेखाऐं प्रतिच्छेद करेगी यदि वे समतलीय है

$\vec{a} - \vec{c}$, $\hat{b}$ & $\vec{d}$ समतलीय है।

$\therefore \quad [\vec{a} - \vec{c},\ \ \vec{b},\ \ \vec{d}] = 0$

$$\begin{vmatrix} 2 & k+1 & -1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad 2(-5) - (k + 1)(-2) - 1(1) = 0$

$\Rightarrow \quad 2(k + 1) = 11$

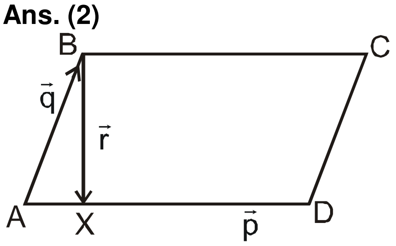
$\Rightarrow \quad k = \dfrac{9}{2}$

16. Let ABCD be a parallelogram such that $\overrightarrow{AB} = \vec{q}$, $\overrightarrow{AD} = \vec{p}$ and $\angle BAD$ be an acute angle. If $\vec{r}$ is the vector that coincides with the altitude directed from the vertex B to the side AD, then is given by :

माना ABCD एक ऐसा समांतर चतुर्भुज है कि $\overrightarrow{AB} = \vec{q}$, $\overrightarrow{AD} = \vec{p}$ तथा $\angle BAD$ एक न्यून कोण है। यदि सदिश $\vec{r}$, शीर्ष B से AD पर खींचे गए लंब के संपाती है, तो निम्न द्वारा प्रदत्त है : **[AIEEE-2012, (4, –1)/120]**

(1) $\vec{r} = 3\vec{q} - \dfrac{3(\vec{p}\,.\,\vec{q})}{(\vec{p}\,.\,\vec{p})}\,\vec{p}$ (2*) $\vec{r} = -\vec{q} + \left(\dfrac{\vec{p}\,.\,\vec{q}}{\vec{p}\,.\,\vec{p}}\right)\vec{p}$ (3) $\vec{r} = \vec{q} - \left(\dfrac{\vec{p}\,.\,\vec{q}}{\vec{p}\,.\,\vec{p}}\right)\vec{p}$ (4) $\vec{r} = -3\vec{q} + \dfrac{3(\vec{p}\,.\,\vec{q})}{(\vec{p}\,.\,\vec{p})}\,\vec{p}$

Sol. **Ans. (2)**



$\overrightarrow{AX} = \dfrac{\vec{p}\,.\,\vec{q}}{|\vec{p}|}\dfrac{\vec{p}}{|\vec{p}|} = \dfrac{\vec{p}\,.\,\vec{q}}{|\vec{p}|^2}\,\vec{p}$

$\overrightarrow{BX} = \overrightarrow{BA} + \overrightarrow{AX}$

$= -\vec{q} + \dfrac{\vec{p}\,.\,\vec{q}}{|\vec{p}|^2}\,\vec{p}$

17. Distance between two parallel planes 2x + y + 2z = 8 and 4x + 2y + 4z + 5 = 0 is

[AIEEE - 2013, (4, –1),360]

दो समान्तर समतलों $2x + y + 2z = 8$ तथा $4x + 2y + 4z + 5 = 0$ के बीच की दूरी है–

[AIEEE - 2013, (4, –1),360]

(1) $\frac{3}{2}$ (2) $\frac{5}{2}$ (3*) $\frac{7}{2}$ (4) $\frac{9}{2}$

Sol. **(3)**

$2x + y + 2z - 8 = 0$ ...(P_1)

$2x + y + 2z + \frac{5}{2} = 0$ ...(P_2)

Distance between P_1 and P_2 = $\left|\frac{-8-\frac{5}{2}}{\sqrt{2^2+1^2+2^2}}\right| = \frac{7}{2}$

Hindi. **(3)**

$2x + y + 2z - 8 = 0$ ...(P_1)

$2x + y + 2z + \frac{5}{2} = 0$ ...(P_2)

P_1 और P_2 के बीच की दूरी = $\left|\frac{-8-\frac{5}{2}}{\sqrt{2^2+1^2+2^2}}\right| = \frac{7}{2}$

18. If the lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k}$ and $\frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$ are coplanar, then k can have

[AIEEE - 2013, (4, –1),360]

(1) any value (2) exactly one value (3*) exactly two values (4) exactly three values

यदि रेखाएं $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k}$ तथा $\frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$ समतलीय है, तो k का

(1) कोई भी मान संभव है। (2) केवल एक मान संभव है। (3) केवल दो मान संभव है। (4) केवल तीन मान संभव है।

[AIEEE - 2013, (4, –1),360]

Sol. **(3)**

$[a - c, b, d] = 0$

$$\begin{vmatrix} 2-1 & 3-4 & 4-5 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow 1(1 + 2k) + (1 + k^2) - (2 - k) = 0$

$\Rightarrow k^2 + 2k + k = 0$

$\Rightarrow k^2 + 3k = 0$

$\Rightarrow k = 0, -3$

Note : If 0 appears in the denominator, then the correct way of representing the equation of straight line is $\frac{x-2}{1} = \frac{y-3}{1}$; $z = 4$

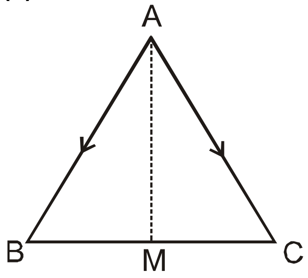
नोट : यदि 0 हर में आता है, तब सरल रेखा को निरूपित करनें का सही तरीका $\frac{x-2}{1} = \frac{y-3}{1}$; $z = 4$ है।

19. If the vectors $\overrightarrow{AB} = 3\hat{i} + 4\hat{k}$ and $\overrightarrow{AC} = 5\hat{i} - 2\hat{j} + 4\hat{k}$ are the sides of a triangle ABC, then the length of the median through A is **[AIEEE - 2013, (4, –¼),360]**

यदि सदिश $\overrightarrow{AB} = 3\hat{i} + 4\hat{k}$ तथा $\overrightarrow{AC} = 5\hat{i} - 2\hat{j} + 4\hat{k}$ एक त्रिभुज ABC की भुजाएँ है, तो A से होकर जाती हुई माध्यिका की लम्बाई है– **[AIEEE - 2013, (4, –¼),360]**

(1) $\sqrt{18}$ (2) $\sqrt{72}$ (3*) $\sqrt{33}$ (4) $\sqrt{45}$

Sol. **(3)**



$\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA} = 0$

$\Rightarrow \quad \overrightarrow{BC} = \overrightarrow{AC}-\overrightarrow{AB}$

$\Rightarrow \quad \overrightarrow{BM} = \dfrac{\overrightarrow{AC}-\overrightarrow{AB}}{2}$

$\Rightarrow \quad \overrightarrow{AB}+\overrightarrow{BM}+\overrightarrow{MA} = 0$

$\Rightarrow \quad \overrightarrow{AB}+\dfrac{\overrightarrow{AC}-\overrightarrow{AB}}{2} = \overrightarrow{AM}$

$\Rightarrow \quad \overrightarrow{AM} = \dfrac{\overrightarrow{AB}+\overrightarrow{AC}}{2} = 4\hat{i}-\hat{j}+4\hat{k}$

$\Rightarrow \quad |\overrightarrow{AM}| = \sqrt{33}$

20. The image of the line $\dfrac{x-1}{3}=\dfrac{y-3}{1}=\dfrac{z-4}{-5}$ in the plane $2x - y + z + 3 = 0$ is the line :

समतल $2x - y + z + 3 = 0$ में रेखा $\dfrac{x-1}{3}=\dfrac{y-3}{1}=\dfrac{z-4}{-5}$ के प्रतिबिम्ब वाली रेखा है :

[JEE(Main) 2014, (4, – 1), 120]

(1) $\dfrac{x-3}{3}=\dfrac{y+5}{1}=\dfrac{z-2}{-5}$ (2) $\dfrac{x-3}{-3}=\dfrac{y+5}{-1}=\dfrac{z-2}{5}$

(3*) $\dfrac{x+3}{3}=\dfrac{y-5}{1}=\dfrac{z-2}{-5}$ (4) $\dfrac{x+3}{-3}=\dfrac{y-5}{-1}=\dfrac{z+2}{5}$

Sol. **Ans. (3)**

$$\frac{x-1}{2}=\frac{y-3}{-1}=\frac{z-4}{1}=\frac{-2(2-3+4+3)}{4+1+1}$$

$$\Rightarrow \quad \frac{x-1}{2}=\frac{y-3}{-1}=\frac{z-4}{1}=-2$$

Image of point (1, 3, 4) on given line in the given plane is (–3, 5, 2)

रेखा पर स्थित बिन्दु (1, 3, 4) का समतल मे प्रतिबिम्ब है (–3, 5, 2)

Line is parallel to given plane रेखा दिये समतल के समान्तर है 3, 1, –5

So, image इसलिए प्रतिबिंब

$$\frac{x+3}{3}=\frac{y-5}{1}=\frac{z-2}{-5}$$

21✎. The angle between the lines whose direction cosines satisfy the equations $\ell + m + n = 0$ and $\ell^2 = m^2 + n^2$ is

[Vector & 3-D] [JEE(Main) 2014, (4, – 1), 120]

दो रेखाऐं जिनके दिक् कोज्या, समीकरणों $\ell + m + n = 0$ तथा $\ell^2 = m^2 + n^2$ को संतुष्ट करते है, के बीच का कोंण है–

[Vector & 3-D] [JEE(Main) 2014, (4, – 1), 120]

(1) $\dfrac{\pi}{6}$ (2) $\dfrac{\pi}{2}$ (3*) $\dfrac{\pi}{3}$ (4) $\dfrac{\pi}{4}$

Sol. **Ans. (3)**

$\ell + m + n = 0$ ....(1)

$\ell^2 = m^2 + n^2$ ....(2)

$\Rightarrow \quad \ell^2 - m^2 - (-\ell - m)^2 = 0$

$\Rightarrow \quad 2m(m + \ell) = 0$

$m = 0$ or $\ell = -m$

so direction ratios are इसलिए द्विक् अनुपात है – 1, 0, 1 और and – 1, 1, 0

$$\therefore \quad \cos\theta = \left(\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right)$$

$$\Rightarrow \cos\theta = \left|\frac{1 + 0 + 0}{\sqrt{2}\ \sqrt{2}}\right| = \frac{1}{2}$$

$$\therefore \theta = \frac{\pi}{3}$$

22. If $\left[\vec{a} \times \vec{b} \quad \vec{b} \times \vec{c} \quad \vec{c} \times \vec{a}\right] = \lambda\left[\vec{a} \quad \vec{b} \quad \vec{c}\right]^2$ then λ is equal to **[JEE(Main) 2014, (4, – 1), 120]**

यदि $\left[\vec{a} \times \vec{b} \quad \vec{b} \times \vec{c} \quad \vec{c} \times \vec{a}\right] = \lambda\left[\vec{a} \quad \vec{b} \quad \vec{c}\right]^2$ है, तो λ बराबर है– **[JEE(Main) 2014, (4, – 1), 120]**

(1) 0 (2*) 1 (3) 2 (4) 3

Sol. **Ans. (2)**

LHS = $[\vec{a} \times \vec{b} \quad \vec{b} \times \vec{c} \quad \vec{c} \times \vec{a}]$

$= [\vec{p} \quad \vec{b} \times \vec{c} \quad \vec{c} \times \vec{a}]$ (where जहाँ $\vec{p} = \vec{a} \times \vec{b}$)

$= \{\vec{p} \times (\vec{b} \times \vec{c})\}.(\vec{c} \times \vec{a})$

$= \{\vec{p} \times (\vec{b} \times \vec{c})\ (\vec{c} \times \vec{a})$

$= \{(\vec{p}.\vec{c})\vec{b} - (\vec{p}.\vec{b})\vec{c}\}.(\vec{c} \times \vec{a})$

$= \{[\vec{a}\,\vec{b}\,\vec{c}] - [\vec{a}\,\vec{b}\,\vec{c}]\}.(\vec{c} \times \vec{a})$ (As $\vec{p} = \vec{a} \times \vec{b}$)

$= [\vec{a}\,\vec{b}\,\vec{c}][\vec{b} \quad \vec{c} \quad \vec{a}] - 0$ $(\because [\vec{a} \quad \vec{b} \quad \vec{b}] = 0)$

$= [\vec{a} \quad \vec{b} \quad \vec{c}]^2 (\because [\vec{b} \quad \vec{c} \quad \vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}])$

RHS = $\lambda\,[\vec{a}\,\vec{b}\,\vec{c}]^2$

$\therefore \lambda = 1$

23. The distance of the point (1,0,2) from the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 16$, is **[JEE(Main) 2015, (4, – ¼), 120]**

(1) $2\sqrt{14}$ (2) 8 (3) $3\sqrt{21}$ (4) 13

रेखा $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ तथा समतल $x - y + z = 16$ के प्रतिच्छेद बिन्दु की, बिन्दु (1, 0, 2) से दूरी है–

[JEE(Main) 2015, (4, – ¼), 120]

(1) $2\sqrt{14}$ (2) 8 (3) $3\sqrt{21}$ (4) 13

Ans. **(4)**

Sol. Point of intersection

$(3\lambda + 2, 4\lambda - 1, 12\lambda + 2)$

$3\lambda + 2 - 4\lambda + 1 + 12\lambda + 2 = 16$

$11\lambda = 11 \quad \Rightarrow \quad \lambda = 1$

(5, 3, 14)

Distance = $\sqrt{16+9+144} = \sqrt{169} = 13$

Hindi. प्रतिच्छेद बिन्दु

$(3\lambda + 2, 4\lambda - 1, 12\lambda + 2)$

$3\lambda + 2 - 4\lambda + 1 + 12\lambda + 2 = 16$

$11\lambda = 11 \qquad \Rightarrow \qquad \lambda = 1$

$(5, 3, 14)$

दुरी = $\sqrt{16+9+144} = \sqrt{169} = 13$

24. The equation of the plane containing the line $2x - 5y + z = 3$, $x + y + 4z = 5$ and parallel to the plane $x + 3y + 6z = 1$, is **[JEE(Main) 2015, (4, – ¼), 120]**

(1) $2x + 6y + 12z = 13$ (2) $x + 3y + 6z = -7$ (3) $x + 3y + 6z = 7$ (4) $2x + 6y + 12z = -13$

रेखा $2x - 5y + z = 3$, $x + y + 4z = 5$ को अंतर्विष्ट करने वाले समतल, जो समतल $x + 3y + 6z = 1$ के समान्तर है, का समीकरण है– **[JEE(Main) 2015, (4, – ¼), 120]**

(1) $2x + 6y + 12z = 13$ (2) $x + 3y + 6z = -7$ (3) $x + 3y + 6z = 7$ (4) $2x + 6y + 12z = -13$

Ans. **(3)**

Sol. Equation of real plane

$2x - 5y + z - 3 + \lambda(x + y + 4z - 5) = 0$

$x(2 + \lambda) + y(\lambda - 5) + z(4\lambda + 1) - 3 - 5\lambda = 0$

$\Rightarrow \quad \frac{\lambda+2}{1} = \frac{\lambda-5}{3} = \frac{4\lambda+1}{6} = -3 + \frac{55}{2}$

$3\lambda + 6 = \lambda - 5$

$2\lambda = -11$

$\lambda = \frac{-11}{2}$

$\Rightarrow$ equation of plane $\frac{-7x}{2} - \frac{21y}{2} - 21z + \frac{49}{2} = 0 \quad \Rightarrow \quad 7x + 21y + 42z - 49 = 0$

$\Rightarrow \quad x + 3y + 6z - 7 = 0$

Hindi. वास्तविक समतल का समीकरण

$2x - 5y + z - 3 + \lambda(x + y + 4z - 5) = 0$

$x(2 + \lambda) + y(\lambda - 5) + z(4\lambda + 1) - 3 - 5\lambda = 0$

$\Rightarrow \quad \frac{\lambda+2}{1} = \frac{\lambda-5}{3} = \frac{4\lambda+1}{6} = -3 + \frac{55}{2}$

$3\lambda + 6 = \lambda - 5$

$2\lambda = -11$

$\lambda = \frac{-11}{2}$

$\Rightarrow$ समतल का समीकरण $\frac{-7x}{2} - \frac{21y}{2} - 21z + \frac{49}{2} = 0 \quad \Rightarrow \quad 7x + 21y + 42z - 49 = 0$

$\Rightarrow \quad x + 3y + 6z - 7 = 0$

25. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three non-zero vectors such that no two of them are collinear and $(\vec{a} \times \vec{b}) \times \vec{c} = \frac{1}{3}|\vec{b}||\vec{c}|\,\vec{a}$. If θ is the angle between vectors $\vec{b}$ and $\vec{c}$, then a value of $\sin\theta$ is

[JEE(Main) 2015, (4, – ¼), 120]

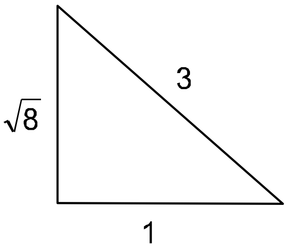
(1) $\frac{2\sqrt{2}}{3}$ (2) $\frac{-\sqrt{2}}{3}$ (3) $\frac{2}{3}$ (4) $\frac{-2\sqrt{3}}{3}$

माना $\vec{a}$, $\vec{b}$ तथा $\vec{c}$ तीन शून्येतर ऐसे सदिश है कि उनमें से कोई दो संरेख नहीं है तथा $(\vec{a}\times\vec{b})\times\vec{c}=\frac{1}{3}|\vec{b}||\vec{c}|\,\vec{a}$ है। यदि सदिशों $\vec{b}$ और $\vec{c}$ के बीच का कोण θ है, तो $\sin\theta$ का एक मान है– **[JEE(Main) 2015, (4, – ¼), 120]**

(1) $\frac{2\sqrt{2}}{3}$ (2) $\frac{-\sqrt{2}}{3}$ (3) $\frac{2}{3}$ (4) $\frac{-2\sqrt{3}}{3}$

Ans. **(1)**

Sol.



$$(\vec{a}\times\vec{b})\times\vec{c}=\frac{1}{3}|b||c|\vec{a}$$

$$-\vec{c}\times(\vec{a}\times\vec{b})$$

$$-(\vec{c}.\vec{b})\vec{a}+(\vec{c}.\vec{a})\vec{b}=\frac{1}{3}|b||c|\vec{a}$$

$$\left(\frac{1}{3}|b||c|+(\vec{c}.\vec{b})\right)\vec{a}=(\vec{c}.\vec{a})\vec{b}$$

Since $\vec{a}$ & $\vec{b}$ are not collinear

$$\vec{c}.\vec{b}+\frac{1}{3}|b|\,|c|=0 \quad \& \quad \vec{c}.\vec{a}=0$$

$$\cos\theta+\frac{1}{3}=0$$

$$\cos\theta=-\frac{1}{3} \quad \Rightarrow \quad \sin\theta=\frac{\sqrt{8}}{3}=\frac{2\sqrt{2}}{3}$$

Aliter :

$$(\vec{a}\times\vec{b})\vec{c}=(\vec{a}.\vec{c})\vec{b}-(\vec{b}.\vec{c})\vec{a}=\frac{1}{3}|\vec{b}||\vec{c}|\vec{a}$$

$$(\vec{a}.\vec{c})\vec{b}=\left(\frac{1}{3}|\vec{b}||\vec{c}|+\vec{b}.\vec{c}\right)\vec{a}$$

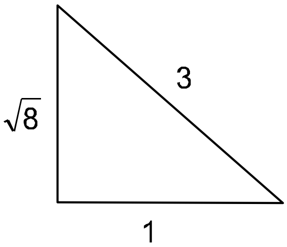
$$\vec{a}.\vec{c}=0 \qquad \& \qquad \frac{1}{3}|\vec{b}||\vec{c}|+\vec{b}.\vec{c}=0$$

$$\frac{1}{3}|\vec{b}||\vec{c}|+|\vec{b}||\vec{c}|\cos\theta=0$$

$$\cos\theta=-\frac{1}{3}$$

$$\sin\theta=\sqrt{1-\frac{1}{9}}=\frac{2\sqrt{2}}{3}$$

Hindi.



$\left(\vec{a}\times\vec{b}\right)\times\vec{c}=\frac{1}{3}|b||c|\vec{a}$

$-\vec{c}\times\left(\vec{a}\times\vec{b}\right)$

$-\left(\vec{c}.\vec{b}\right)\vec{a}+\left(\vec{c}.\vec{a}\right)\vec{b}=\frac{1}{3}|b||c|\vec{a}$

$\left(\frac{1}{3}|b||c|+\left(\vec{c}.\vec{b}\right)\right)\vec{a}=\left(\vec{c}.\vec{a}\right)\vec{b}$

अतः $\vec{a}$ और $\vec{b}$ संरेखीय नहीं है।

$\vec{c}.\vec{b}+\frac{1}{3}|b|\,|c|=0$ & $\vec{c}.\vec{a}=0$

$\cos\theta+\frac{1}{3}=0$

$\cos\theta=-\frac{1}{3} \quad \Rightarrow \quad \sin\theta=\frac{\sqrt{8}}{3}=\frac{2\sqrt{2}}{3}$

वैकल्पिक : $\left(\vec{a}\times\vec{b}\right)\vec{c}=\left(\vec{a}.\vec{c}\right)\vec{b}-\left(\vec{b}.\vec{c}\right)\vec{a}=\frac{1}{3}|\vec{b}||\vec{c}|\vec{a}$

$\left(\vec{a}.\vec{c}\right)\vec{b}=\left(\frac{1}{3}|\vec{b}||\vec{c}|+\vec{b}.\vec{c}\right)\vec{a}$

$\vec{a}.\vec{c}=0$ & $\frac{1}{3}|\vec{b}||\vec{c}|+\vec{b}.\vec{c}=0$

$\frac{1}{3}|\vec{b}||\vec{c}|+|\vec{b}||\vec{c}|\cos\theta=0$

$\cos\theta=-\frac{1}{3}$

$\sin\theta=\sqrt{1-\frac{1}{9}}=\frac{2\sqrt{2}}{3}$

26. If the line, $\frac{x-3}{2}=\frac{y+2}{-1}=\frac{z+4}{3}$ lies in the plane, $lx+my-z=9$, then l^2+m^2 is equal to

[JEE(Main) 2016, (4, – 1), 120]

यदि रेखा $\frac{x-3}{2}=\frac{y+2}{-1}=\frac{z+4}{3}$, समतल $lx+my-z=9$ में स्थित है, तो l^2+m^2 बराबर है–

(1) 18 (2) 5 (3) 2 (4) 26

Ans. **(3)**

Sol. (i) (3, – 2, –4) lies on the plane (समतल पर स्थित है)

$\therefore 3\ell-2m+4=9 \quad \Rightarrow 3\ell-2m=5$ .... (i)

(ii) $2\ell-m-3=0 \quad \Rightarrow 2\ell-m=3$ .... (ii)

from (i) and (ii) (i) तथा (ii) से

$\ell=1$ and (और) $m=-1$

27. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three unit vectors such that $\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \frac{\sqrt{3}}{2}(\vec{b} + \vec{c})$. If $\vec{b}$ is not parallel to $\vec{c}$, then the angle between $\vec{a}$ and $\vec{b}$ is **[JEE(Main) 2016, (4, – 1), 120]**

माना $\vec{a}, \vec{b}$ तथा $\vec{c}$ तीन ऐसे मात्रक सदिा है कि $\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \frac{\sqrt{3}}{2}(\vec{b} + \vec{c})$ है। यदि $\vec{b}$, $\vec{c}$ के समान्तर नहीं है, तो $\vec{a}$ तथा $\vec{b}$ के बीच का कोण है–

(1) $\frac{\pi}{2}$ (2) $\frac{2\pi}{3}$ (3) $\frac{5\pi}{6}$ (4) $\frac{3\pi}{4}$

Ans. **(3)**

Sol. $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\sqrt{3}}{2}(\vec{b} + \vec{c})$

$(\vec{a} . \vec{c}) - (\vec{a} . \vec{b})\vec{c} = \frac{\sqrt{3}}{2}\vec{b} + \frac{\sqrt{3}}{2}\vec{c}$

Hence (अत:) $\vec{a} . \vec{c} = \frac{\sqrt{3}}{2}$ and (और) $\vec{a} . \vec{b} = -\frac{\sqrt{3}}{2}$

$\vec{a} . \vec{b} = -\frac{\sqrt{3}}{2}$

$\cos\theta = -\frac{\sqrt{3}}{2}$

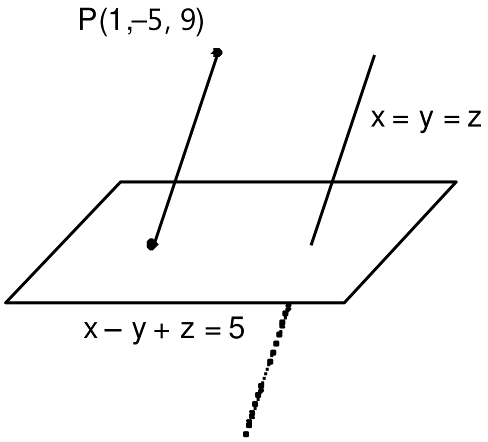
$\theta = \frac{5\pi}{6}$

28. The distance of the point (1, –5, 9) from the plane $x - y + z = 5$ measured along the line $x = y = z$ is **[JEE(Main) 2016, (4, – 1), 120]**

बिन्दु (1, –5, 9) की समतल $x - y + z = 5$ से वह दूरी जो रेखा $x = y = z$ की दिशा में मापी गई है, है–

(1) $10\sqrt{3}$ (2) $\frac{10}{\sqrt{3}}$ (3) $\frac{20}{3}$ (4) $3\sqrt{10}$

Ans. **(1)**



Sol.

Equation of line PQ : $\frac{x-1}{1} = \frac{y+5}{1} = \frac{z-9}{1} = \lambda$

रेखा PQ का समीकरण है $\frac{x-1}{1} = \frac{y+5}{1} = \frac{z-9}{1} = \lambda$

$\therefore$ Q can be taken as $(\lambda + 1, \lambda - 5, \lambda + 9)$

$\therefore$ Q को $(\lambda + 1, \lambda - 5, \lambda + 9)$ माना जा सकता है।

As Q lies on plane $x - y + z = 5$
जैसा कि Q समतल $x - y + z = 5$ पर स्थित है

$\therefore$ $(\lambda + 1) - (\lambda - 5) + (\lambda + 9) = 5$
$\lambda = -10 \quad \Rightarrow \quad Q(-9, -15, -1)$

$\therefore$ Required distance PQ = $\sqrt{(1+9)^2+(-5+15)^2+(9+1)^2} = \sqrt{100+100+100} = 10\sqrt{3}$

$\therefore$ अभीट दूरी PQ = $\sqrt{(1+9)^2+(-5+15)^2+(9+1)^2} = \sqrt{100+100+100} = 10\sqrt{3}$

29. If the image of the point P(1, –2, 3) in the plane, $2x + 3y - 4z + 22 = 0$ measured parallel to the line, $\frac{x}{1} = \frac{y}{4} = \frac{z}{5}$ is Q, then PQ is equal to : **[JEE(Main) 2017, (4, – ¼), 120]**

यदि बिंदु P(1, –2, 3) का समतल $2x + 3y - 4z + 22 = 0$ में यह प्रतिबिम्ब की रेखा $\frac{x}{1} = \frac{y}{4} = \frac{z}{5}$ के समांतर है Q है, तो PQ बराबर है :

(1) $3\sqrt{5}$ (2) $2\sqrt{42}$ (3) $\sqrt{42}$ (4) $6\sqrt{5}$

Ans. **(2)**

Sol. Let R be the point of intersection of plane and line passing through P and parallel to given line.
समतल तथा P से गुजरने वाली रेखा और दी गई रेखा के समान्तर रेखा का प्रतिच्छेद बिन्दु माना R है।
So, अतः R is है। $(1 + \lambda, -2 + 4\lambda, 3 + 5\lambda)$
substituting co-ordinates of R in plane
समतल में R के निर्दांक प्रतिस्थापित करने पर
$2 + 2\lambda - 6 + 12\lambda - 12 - 20\lambda + 22 = 0 \quad \Rightarrow 6\lambda = 6 \Rightarrow \lambda = 1$
So अतः, R is (2,2,8)

Hence इस प्रकार PR = $\sqrt{1+16+25} = \sqrt{42}$

So इसलिए, PQ = $2\sqrt{42}$

30. The distance of the point (1, 3, – 7) from the plane passing through the point (1, –1, –1), having normal perpendicular to both the lines $\frac{x-1}{1} = \frac{x+2}{-2} = \frac{x-4}{3}$ and $\frac{x-2}{2} = \frac{y+1}{-1} = \frac{z+7}{-1}$, is

[JEE(Main) 2017, (4, – ¼), 120]

एक समतल जो बिन्दु (1, –1, –1) से होकर जाता है तथा जिसका अभिलम्ब दोनों रेखाओं $\frac{x-1}{1} = \frac{x+2}{-2} = \frac{x-4}{3}$ तथा $\frac{x-2}{2} = \frac{y+1}{-1} = \frac{z+7}{-1}$ पर लम्ब है, को बिन्दु (1, 3, – 7) से दुरी है :

(1) $\frac{20}{\sqrt{74}}$ (2) $\frac{10}{\sqrt{83}}$ (3) $\frac{5}{\sqrt{83}}$ (4) $\frac{10}{\sqrt{74}}$

Ans. **(2)**

Sol. Let the plane be माना समतल
$a(x - 1) + b(y + 1) + c(z + 1) = 0$
$a - 2b + 3c = 0$
$2a - b - c = 0$
$\frac{a}{5} = \frac{b}{7} = \frac{c}{3}$
$5(x - 1) + 7(y + 1) + 3(z + 1) = 0$
$5x + 7y + 3z + 5 = 0$
P(1, 3, –7)

$$d = \left|\frac{5+21-21+5}{\sqrt{25+49+9}}\right| = \left|\frac{10}{\sqrt{83}}\right|$$

31. Let $\vec{a} = 2\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b} = \hat{i}+\hat{j}$. Let $\vec{c}$ be a vector such that $|\vec{c}-\vec{a}|$ = 3, $\left|(\vec{a}\times\vec{b})\times\vec{c}\right|$ = 3 and the angle between $\vec{c}$ and $\vec{a}\times\vec{b}$ be 30º. Then $\vec{a}.\vec{c}$ is equal to **[JEE(Main) 2017, (4, – 1), 120]**

माना $\vec{a} = 2\hat{i}+\hat{j}-2\hat{k}$ तथा $\vec{b} = \hat{i}+\hat{j}$ है। माना $\vec{c}$ एक ऐसा सदिश है कि $|\vec{c}-\vec{a}|$ = 3, $\left|(\vec{a}\times\vec{b})\times\vec{c}\right|$ = 3 तथा $\vec{c}$ और $\vec{a}\times\vec{b}$ के बीच के कोण 30º है, तो $\vec{a}.\vec{c}$ बराबर है–

(1) $\frac{25}{8}$ (2) 2 (3) 5 (4) $\frac{1}{8}$

Ans. **(2)**

Sol. $\vec{a} = 2\hat{i}+\hat{j}-2\hat{k},\ \vec{b} = \hat{i}+\hat{j}, |\vec{c}-\vec{a}| = 3$

$|(\vec{a}\times\vec{b})\times\vec{c}| = 3$, $\vec{c} \wedge \vec{a}\times\vec{b} = \frac{\pi}{6}$

Now अब $|\vec{a}\times\vec{b}||\vec{c}|\ \sin30^0 = 3,\ |\vec{a}\times\vec{b}||\vec{c}| = 6$

$\Rightarrow$ $|\vec{a}||\vec{b}||\vec{c}|\sin\theta = 6,$ $\theta = \vec{a}\wedge\vec{b}$

$|\vec{a}| = 3,$ $|\vec{b}| = \sqrt{2}$ $\theta = \cos^{-1}\left(\frac{2+1}{3\sqrt{2}}\right) = \frac{\pi}{4}$

$|\vec{c}| = \frac{6}{3\sqrt{2}}.\sqrt{2} = 2$

$|\vec{c}-\vec{a}| = 3$

Squaring, we get वर्ग करने पर $|\vec{c}|^2 - 2\vec{a}.\vec{c} + |\vec{a}|^2 = 9$ $\Rightarrow$ $\vec{a}.\vec{c} = \frac{|\vec{c}|^2}{2} = 2$

32. If L_1 is the line of intersection of the planes 2x – 2y + 3z – 2 = 0, x – y + z + 1 = 0 and L_2 is the line of intersection of the planes x + 2y – z – 3 = 0, 3x – y + 2z – 1 = 0, then the distance of the origin from the plane, containing the lines L_1 and L_2, is : **[JEE(Main) 2018, (4, –1), 120]**

यदि समतलों 2x – 2y + 3z – 2 = 0, x – y + z + 1 = 0 की परिच्छेदी रेखा L_1 है तथा समतलों x + 2y – z – 3 = 0, 3x – y + 2z – 1 = 0 की परिच्छेदी रेखा L_2 है , तो मूल बिंदु की दूरी उस सममतल से जो रेखाओं L_1 और L_2 का अंतर्विष्ट करता है, है :

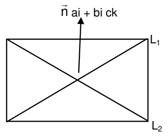
(1) $\frac{1}{2\sqrt{2}}$ (2) $\frac{1}{\sqrt{2}}$ (3) $\frac{1}{4\sqrt{2}}$ (4*) $\frac{1}{3\sqrt{2}}$

Sol. **(4)**

$L_1 : L_1 : \begin{cases} 2x-2y+3z-2=0 \\ x-y+z+1=0 \end{cases}$

Let a point on L_1(0, 5, 4) and dr; s of L_1 be a, b, c

माना बिन्दु L_1(0, 5, 4) है और L_1 के दिक् अनुपात a, b, c है



$2a_1 + 2b_1 + 3c_1 = 0$

$a_1 + b_1 + c_1 = 0$

$\frac{a_1}{1} = \frac{b_1}{1} = \frac{c_1}{0}$

so dr's of L_2 be a_2, b_2, c_2

dr's of L_2 can be $3, -5, -7$

so dr's of normal to the plane ca be $a + b + oc = 0$

$3a - 5b - 7c = 0$

$\frac{a}{-7} = \frac{b}{7} = \frac{c}{-8}$

equation req. plane $7x - 7(y-5) + 8(z-4) = 0$

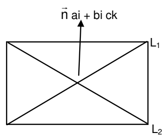
$7x - 7y + 8z + 3 = 0$

so req. distance = $\frac{3}{\sqrt{49+49+64}} = \frac{3}{\sqrt{162}} = \frac{3}{9\sqrt{2}} = \frac{1}{3\sqrt{2}}$

Hindi (4)

$L_1 : L_1 : \begin{cases} 2x - 2y + 3z - 2 = 0 \\ x - y + z + 1 = 0 \end{cases}$

माना बिन्दु $L_1(0, 5, 4)$ है और L_1 के दिक् अनुपात a, b, c है



$2a_1 + 2b_1 + 3c_1 = 0$

$a_1 + b_1 + c_1 = 0$

$\frac{a_1}{1} = \frac{b_1}{1} = \frac{c_1}{0}$

इसलिए L_2 के दिक् अनुपात a_2, b_2, c_2

L_2 के दिक् अनुपात हो सकते है 3, –5, –7

समतल के अभिलम्ब के दिक् अनुपात $a + b + oc = 0$

$3a - 5b - 7c = 0$

$$\frac{a}{-7} = \frac{b}{7} = \frac{c}{-8}$$

समतल का समीकरण $7x - 7(y-5) + 8(z-4) = 0$

$7x - 7y + 8z + 3 = 0$

अभीष्ट दूरी $= \frac{3}{\sqrt{49+49+64}} = \frac{3}{\sqrt{162}} = \frac{3}{9\sqrt{2}} = \frac{1}{3\sqrt{2}}$

33. Let $\vec{u}$ be a vector coplanar with the vectors $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{j} + \hat{k}$. If $\vec{u}$ is perpendicular to $\vec{a}$ and $\vec{u}.\vec{b} = 24$, then $|\vec{u}|^2$ is equal to : **[JEE(Main) 2018, (4, –1), 120]**

माना $\vec{u}$ एक ऐसा सदिश है जो सदिशों $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ तथा $\vec{b} = \hat{j} + \hat{k}$ के साथ समतलीय है। यदि $\vec{u}$, $\vec{a}$ पर लंबवत् है तथा $\vec{u}.\vec{b} = 24$ है, तो $|\vec{u}|^2$ बराबर है :

(1) 256 (2) 84 (3*) 336 (4) 315

Sol. (3)

$\vec{u} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{u}.\vec{a} = 0 \Rightarrow 2x + 3y - z = 0$ ....(i)

$\vec{u}.\vec{b} = 24 \Rightarrow y + z = 24$ ....(ii)

$$[\vec{u}\ \vec{a}\ \vec{b}] = 0 \Rightarrow \begin{vmatrix} x & y & z \\ 2 & 3 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

$\Rightarrow 4x - 2y + 2z = 0$

$2x - y + z = 0$ ....(iii)

(2) + (3)

$2x + 2z = 24$

$x + z = 12$ ....(iv)

Now अतः $24 - 2z + 3(24 - z) - z = 0$

$96 = 6z$

$z = 16 \Rightarrow x = -4 \Rightarrow y = 8$

$\vec{u} = -4\vec{i} + 8\vec{j} + 16\hat{k}$

$|\vec{u}|^2 = 16 + 64 + 256 = 336$

34. The length of the projection of the line segment joining the points (5, – 1, 4) and (4,–1,3) on the plane , $x + y + z = 7$ is : **[JEE(Main) 2018, (4, –1), 120]**

बिंदुओं (5, – 1, 4) तथा (4,–1,3) को मिलाने वाले रेखाखण्ड का समतल $x + y + z = 7$ पर डाले गए प्रक्षेप की लम्बाई है :

(1) $\frac{1}{3}$ (2*) $\sqrt{\frac{2}{3}}$ (3) $\frac{2}{\sqrt{3}}$ (4) $\frac{2}{3}$

Sol. (2)

A (5, –1, 4)

B (4,–1,3)

AB = $\sqrt{2}$

Direction ratio of AB < 1,0,1>

AB के दिक्अनुपात < 1,0,1>

Angle between line AB and plane is $\theta \Rightarrow \sin\theta = \frac{2}{\sqrt{6}}$

रेखा AB का प्रक्षेप $\theta \Rightarrow \sin\theta = \frac{2}{\sqrt{6}}$ $\Rightarrow \cos\theta = \frac{1}{\sqrt{3}}$

Projection of AB on plane = AB $\cos\theta$ = $\sqrt{\frac{2}{3}}$

35. If the lines $x = ay + b$, $z = cy + d$ and $x = a'z + b'$, $y = c'z + d'$ are perpendicular, then :

यदि रेखाएँ $x = ay + b$, $z = cy + d$ तथा $x = a'z + b'$, $y = c'z + d'$ लम्बवत हैं, तो–

[JEE(Main) 2019, Online (09-01-19),P-2 (4, – 1), 120]

(1) $ab' + bc' + 1 = 0$ (2) $bb' + cc' + 1 = 0$

(3) $cc' + a + a' = 0$ (4) $aa' + c + c' = 0$

Vector 3D XII E,

Ans. (4)

Sol. Given lines $\frac{x-b}{a}=\frac{y}{1}=\frac{z-d}{c}$ and $\frac{x-b'}{a'}=\frac{y-d'}{c'}=\frac{z}{1}$

which are $\perp$ to each other

$\Rightarrow$ $aa' + c' + c = 0$

Hindi. दी गई रेखाऐं $\frac{x-b}{a}=\frac{y}{1}=\frac{z-d}{c}$ तथा $\frac{x-b'}{a'}=\frac{y-d'}{c'}=\frac{z}{1}$

एक दूसरे के लम्बवत

$\Rightarrow$ $aa' + c' + c = 0$

36. Let $\vec{a}=\hat{i}+\hat{j}+\sqrt{2}\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+\sqrt{2}\hat{k}$ and $\vec{c}=5\hat{i}+\hat{j}+\sqrt{2}\hat{k}$ be three vectors such that the projection vector of $\vec{b}$ on $\vec{a}$ is $\vec{a}$. If $\vec{a}+\vec{b}$ is perpendicular to $\vec{c}$, then $|\vec{b}|$ is equal to : Vector 3D XII T,

[JEE(Main) 2019, Online (09-01-19),P-2 (4, – 1), 120]

माना $\vec{a}=\hat{i}+\hat{j}+\sqrt{2}\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+\sqrt{2}\hat{k}$ और $\vec{c}=5\hat{i}+\hat{j}+\sqrt{2}\hat{k}$ तीन सदिश इस प्रकार है कि सदिश $\vec{b}$ का $\vec{a}$ पर प्रक्षेप सदिश $\vec{a}$ है। यदि $\vec{a}+\vec{b}$, सदिश $\vec{c}$ के लम्बवत है तब $|\vec{b}|$ बराबर है–

(1) $\sqrt{22}$ (2) 4 (3) $\sqrt{32}$ (4) 6

Ans. (4)

Sol. $(\vec{b}\,.\,\hat{a})\hat{a}=|\vec{a}|\,\hat{a}$

$(\vec{b}-\hat{a})=|\vec{a}|$

$\frac{b_1+b_2+2}{2}=2$

$\Rightarrow$ $b_1 + b_2 = 2$ ......(1)

$\vec{a}-\vec{b}\ \perp\ \vec{c}$

$\Rightarrow$ $5(b_1 + 1) + 1(b_2 + 1) + \sqrt{2}\,(2\sqrt{2}) = 0$

$\Rightarrow$ $5b_1 + b_2 + 10 = 0$ ......(2)

from (1) and (2) $b_1 = -3$ and $b_2 = 5$ (समीकरण (1) व (2) से)

$\Rightarrow$ $|\vec{b}| = \sqrt{9+25+2} = 6$

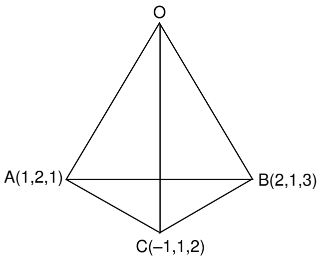
37. A tetrahedron has vertices P(1,2,1), Q(2,1,3), R(–1,1,2) and O(0,0,0) the angle between the faces OPQ and PQR is :

एक चतुष्फलक (tetrahedron) के शीर्ष P(1,2,1), Q(2,1,3), R(–1,1,2) तथा O(0,0,0) है। फलक OPQ तथा PQR के बीच का कोण है–**[JEE(Main) 2019, Online (12-01-19),P-1 (4, – 1), 120]**

(1) $\cos^{-1}\left(\frac{19}{35}\right)$ (2) $\cos^{-1}\left(\frac{7}{31}\right)$ (3) $\cos^{-1}\left(\frac{17}{31}\right)$ (4) $\cos^{-1}\left(\frac{9}{35}\right)$

Ans. **(1)**

Sol.



Vector perpendicular to face OAB = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 5\hat{i} - \hat{j} - 3\hat{k}$

फलक OAB के लम्बवत् सदिश = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 5\hat{i} - \hat{j} - 3\hat{k}$

Vector perpendicular to face ABC = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix} = \hat{i} - 5\hat{j} - 2\hat{k}$

फलक ABC के लम्बवत् सदिश = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix} = \hat{i} - 5\hat{j} - 2\hat{k}$

Angle between two faces $\cos\theta = \left|\frac{5+5+9}{\sqrt{35}\sqrt{35}}\right| = \frac{19}{35}$

दोनों फलकों के मध्य कोण $\cos\theta = \left|\frac{5+5+9}{\sqrt{35}\sqrt{35}}\right| = \frac{19}{35}$

$\Rightarrow \theta = \cos^{-1}\left(\frac{19}{35}\right)$

38. Let S be the set of all real values of λ such that a plane passing through the points $(-\lambda^2, 1, 1)$, $(1, -\lambda^2, 1)$ and $(1, 1, -\lambda^2)$ also passes through the point (–1, –1, 1). Then S is equal to -

यदि λ के उन सभी वास्तविक मानों, जिनके लिए बिन्दुओं $(-\lambda^2, 1, 1), (1, -\lambda^2, 1)$ तथा $(1, 1, -\lambda^2)$ से होकर जाने वाला एक समतल, बिन्दु $(-1, -1, 1)$ से भी होकर जाता है, का समुच्चय S है, तो S बराबर है–

[JEE(Main) 2019, Online (12-01-19),P-2 (4, – 1), 120]

(1) $\{1, -1\}$ (2) $\{\sqrt{3}\}$ (3) $\{\sqrt{3}, -\sqrt{3}\}$ (4) $\{3, -3\}$

Ans. (3)

Sol. $$\begin{vmatrix} -\lambda^2+1 & 2 & 0 \\ 2 & -\lambda^2+1 & 0 \\ 2 & 2 & -\lambda^2-1 \end{vmatrix} = 0 \Rightarrow -(\lambda^2+1).\{(1-\lambda^2)^2-4\} = 0 \Rightarrow \lambda^2 - 1 = \pm 2$$

$$\Rightarrow \lambda^2 = 3 \Rightarrow \lambda = \pm\sqrt{3}$$

39. The magnitude of the projection of the vector $2\hat{i}+3\hat{j}+\hat{k}$ on the vector perpendicular to the plane containing the vectors $\hat{i}+\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}+3\hat{k}$, is :

सदिश $2\hat{i}+3\hat{j}+\hat{k}$ के सदिशों $\hat{i}+\hat{j}+\hat{k}$ तथा $\hat{i}+2\hat{j}+3\hat{k}$ को अंतर्विष्ट करने वाले समतल के लंबवर्तीय सदिश पर प्रक्षेप का परिमाण है–

(1) $3\sqrt{6}$ (2) $\sqrt{6}$ (3) $\sqrt{\frac{3}{2}}$ (4) $\frac{\sqrt{3}}{2}$

[JEE(Main) 2019, Online (08-04-19),P-1 (4, – 1), 120][Vector & 3-D]

Ans. (3)

Sol. Normal vector to the plane containing $\hat{i}+\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}+3\hat{k}$ is

$$\vec{n} = (\hat{i}+\hat{j}+\hat{k}) \times (\hat{i}+2\hat{j}+3\hat{k})$$

$$\vec{n} = \hat{i}-2\hat{j}+\hat{k}$$

projection of $(2\hat{i}+3\hat{j}+\hat{k})$ on $\vec{n}$

$$= \left|\frac{(2\hat{i}+3\hat{j}+\hat{k}).(\hat{i}-2\hat{j}+\hat{k})}{\sqrt{1+4+1}}\right|$$

$$= \frac{3}{\sqrt{6}} = \sqrt{\frac{3}{2}}$$

Hindi. $\hat{i}+\hat{j}+\hat{k}$ तथा $\hat{i}+2\hat{j}+3\hat{k}$ को समाहित करने वाले समतल के लम्बवत सदिश–

$$\vec{n} = (\hat{i}+\hat{j}+\hat{k}) \times (\hat{i}+2\hat{j}+3\hat{k})$$

$$\vec{n} = \hat{i}-2\hat{j}+\hat{k}$$

$(2\hat{i}+3\hat{j}+\hat{k})$ का $\vec{n}$ पर प्रक्षेप

$$= \left|\frac{(2\hat{i}+3\hat{j}+\hat{k}).(\hat{i}-2\hat{j}+\hat{k})}{\sqrt{1+4+1}}\right|$$

$$= \frac{3}{\sqrt{6}} = \sqrt{\frac{3}{2}}$$

40. If the volume of parallelepiped formed by the vectors $\hat{i}+\lambda\hat{j}+\hat{k}, \hat{j}+\lambda\hat{k}$ and $\lambda\hat{i} + \hat{k}$ is minimum , then λ is equal to :

यदि सदिश $\hat{i}+\lambda\hat{j}+\hat{k}, \hat{j}+\lambda\hat{k}$ तथा $\lambda\hat{i} + \hat{k}$ द्वारा बनाये गये समान्तर षट्फलक का आयतन न्यूनतम है, तो बराबर है

(1) $\sqrt{3}$ (2) $-\sqrt{3}$ (3) $\frac{1}{\sqrt{3}}$ (4) $-\frac{1}{\sqrt{3}}$

Ans. **(3)** **[JEE(Main) 2019, Online (12-04-19),P-1 (4, – 1), 120]** **[Vector]**

Sol. $V = \left[\vec{a}\,\vec{b}\,\vec{c}\right] = \begin{vmatrix} 1 & \lambda & 1 \\ 0 & 1 & \lambda \\ \lambda & 0 & 1 \end{vmatrix} = 1 - \lambda(-\lambda^2) + 1.(0-\lambda) = \lambda^3 - \lambda + 1$

Whose minimum value occur at $\lambda = \frac{1}{\sqrt{3}}$

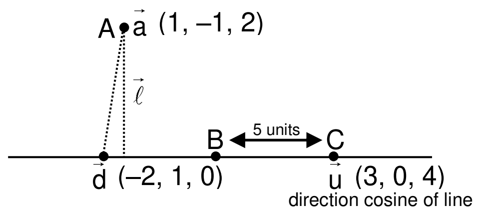
41. The vertices B and C of a ΔABC lie on the line, $\frac{x+2}{3} = \frac{y-1}{0} = \frac{2}{4}$ such that BC = 5 units. Then the area (in sq. units) of this triangle, given that the point A(1, –1, 2), is :

ΔABC के शीर्ष B तथा C रेखा $\frac{x+2}{3} = \frac{y-1}{0} = \frac{2}{4}$ पर स्थित है तथा BC = 5 इकाई है। यदि दिया है कि बिन्दु A(1, –1, 2) है, तो इस त्रिभुज का क्षेत्रफल (वर्ग इकाईयों में) है–

(1) $5\sqrt{17}$ (2) 6 (3) $\sqrt{34}$ (4) $2\sqrt{34}$

Ans. **(3)** **[JEE(Main) 2019, Online (09-04-19),P-2 (4, – 1), 120]** **[Vector & 3-D]**

Sol.



$$|\vec{\ell}| = \left|(\vec{a}-\vec{d})\times u\right|$$

$$|\vec{\ell}| = \left|\left(3\hat{i} - 2\hat{j} + 2k\right) \times \left(\frac{3}{5}\hat{i} + \frac{4}{5}k\right)\right|$$

$$= \left|-\frac{8}{5}\hat{i} + \frac{6}{5}\hat{j} + \frac{6}{5}k\right|$$

$$= \sqrt{\frac{136}{25}}$$

Area of triangle = $\frac{1}{2} \times 5 \times \sqrt{\frac{136}{25}} = \sqrt{34}$ units^2

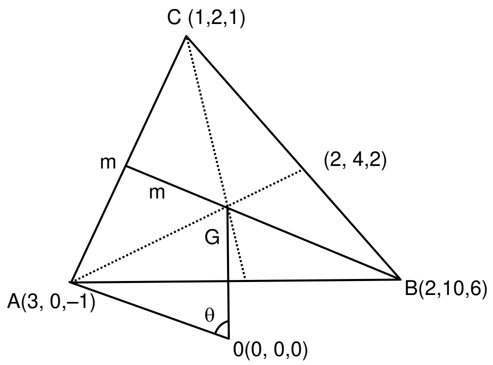
42. Let A(3, 0, –1), B(2, 10, 6) and C(1,2,1) be the vertices of a triangle and M be the midpoint of AC. If G divides BM in the ratio, 2 : 1, then cos($\angle$GOA) (O being the origin) is equal to :

माना एक त्रिभुज के शीर्ष बिन्दु A(3, 0, –1), B(2, 10, 6) तथा C(1,2,1) हैं तथा AC का मध्यबिन्दु M है। यदि G, BM को 2 : 1 के अनुपात में विभाजित करता है, तो cos($\angle$GOA) (O मूलबिन्दु हैं) बराबर है :

(1) $\frac{1}{6\sqrt{10}}$ (2) $\frac{1}{2\sqrt{15}}$ (3) $\frac{1}{\sqrt{15}}$ (4) $\frac{1}{\sqrt{30}}$

Ans. **(3)** **[JEE(Main) 2019, Online (10-04-19),P-1 (4, – 1), 120] [Vector and 3D]**

Sol. A (3,0–1,), B (2,10,6) and C (1,2,1)



G is centroid of Δ from given information .

$$\overrightarrow{OA} . \overrightarrow{OG} = \left|\overrightarrow{OA}\right| \left|\overrightarrow{OG}\right| \cos\theta \quad \Rightarrow \quad \left(2\hat{i} + 4\hat{j} + 2\hat{k}\right) . \left(3\hat{i} - \hat{k}\right) = 2\sqrt{6} . \sqrt{10} \cos\theta$$

$$\Rightarrow \quad 6 - 2 = 2\sqrt{3}.\sqrt{5}.2 \cos\theta$$

$$\cos\theta = \frac{1}{\sqrt{15}}$$

43. A perpendicular is drawn from a point on the line $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z}{1}$ to the plane x + y + z = 3 such that the foot of the perpendicular Q also lies on the plane x – y + z = 3. Then the co-ordinates of Q are :

रेखा $\frac{x-1}{2}=\frac{y+1}{-1}=\frac{z}{1}$ के एक बिन्दु से समतल $x + y + z = 3$ पर एक लम्ब इस प्रकार डाला गया कि इसका लम्बपाद Q, समतल $x - y + z = 3$ पर भी स्थित है। तो Q के निर्देशांक है–

(1) (4, 0, –1) (2) (2, 0, 1) (3) (–1, 0, 4) (4) (1, 0, 2)

Ans. **(2)** **[JEE(Main) 2019, Online (10-04-19),P-2 (4, – 1), 120] [3D]**

Sol. $\frac{x-1}{2}=\frac{y+1}{-1}=\frac{z}{1}=\lambda$

$P(2\lambda + 1, -\lambda - 1, \lambda)$

foot of perpendicular

$$\frac{x-(2\lambda+1)}{1}=\frac{y+(\lambda+1)}{1}=\frac{z-\lambda}{1}=\frac{-(2\lambda+1-\lambda-1+\lambda-3)}{3}$$

$$\frac{x-(2\lambda+1)}{1}=\frac{y+\lambda+1}{1}=\frac{z-\lambda}{1}=\frac{-(2\lambda-3)}{3} \quad \Rightarrow \quad x = 2\lambda + 1 - -\frac{(2\lambda-3)}{3}=\frac{4\lambda+6}{3}$$

$$\Rightarrow \quad y = -\lambda - 1 - -\frac{(2\lambda-3)}{3}=\frac{-3\lambda-3-2\lambda+3}{3}=-\frac{5\lambda}{3}$$

$$\Rightarrow \quad z = \lambda - \frac{(2\lambda-3)}{3}=\frac{\lambda+3}{3}$$

$\therefore$ point P is $\left(\frac{4\lambda+6}{3},\frac{-5\lambda}{3},\frac{\lambda+3}{3}\right)$

It lies on $x - y + z = 3$

$$\frac{4\lambda+6}{3}+\frac{5\lambda}{3}+\frac{\lambda+3}{3}=3 \Rightarrow 10\lambda + 9 = 9 \Rightarrow \lambda = 0$$

$\therefore$ point P becomes (2, 0, 1) $\Rightarrow$ (4) option is correct

44. A vector $\vec{a}=\alpha\hat{i}+2\hat{j}+\beta\hat{k}$ $(\alpha, \beta, \in R)$ lies in the plane of the vectors, $\vec{b}=\hat{i}+\hat{j}$ and $\vec{c}=\hat{i}-\hat{j}+4\hat{k}$.

If $\vec{a}$ bisects the angle between $\vec{b}$ and $\vec{c}$, then: **[JEE(Main) 2020, Online (07-01-20),P-1 (4, – 1), 120]**

एक सदिश $\vec{a}=\alpha\hat{i}+2\hat{j}+\beta\hat{k}$ $(\alpha, \beta, \in R)$ उस समतल में, जिसमें दोनों सदिश $\vec{b}=\hat{i}+\hat{j}$ तथा $\vec{c}=\hat{i}-\hat{j}+4\hat{k}$ स्थित हैं, यदि $\vec{a}$ सदिशों $\vec{b}$ और $\vec{c}$ के बीच के कोण को समद्विभाजित करता है, तो :

(1) $\vec{a}\,.\,\hat{k} + 4 = 0$ (2) $\vec{a}\,.\,\hat{k} + 2 = 0$ (3) $\vec{a}\,.\,\hat{i} + 1 = 0$ (4) $\vec{a}\,.\,\hat{i} + 3 = 0$

Ans. **(2)**

Sol. angle bisector can be कोण अर्द्धक हो सकता है $\vec{a}=\lambda(\hat{b}+\hat{c})$ or या $\vec{a}=\mu(\hat{b}-\hat{c})$

$\vec{a} = \lambda\left(\frac{\hat{i}+\hat{j}}{\sqrt{2}} + \frac{\hat{i}+\hat{j}+4\hat{k}}{3\sqrt{2}}\right) = \frac{\lambda}{3\sqrt{2}}\left[3\hat{i}+3\hat{j}+\hat{i}-\hat{j}+4\hat{k}\right] = \frac{\lambda}{3\sqrt{2}}\left[4\hat{i}+2\hat{j}+4\hat{k}\right]$

Compare with से तुलना करने पर $\vec{a} = \alpha\hat{i}+2\hat{j}+\beta\hat{k}$

$\frac{2\lambda}{3\sqrt{2}} = 2 \Rightarrow \lambda = 3\sqrt{2}$

$\vec{a} = 4\hat{i}+2\hat{j}+4\hat{k}$

Not in option so now consider माना कि $\vec{a} = \mu\left(\frac{\hat{i}+\hat{j}}{\sqrt{2}} - \frac{\hat{i}-\hat{j}+4\hat{k}}{3\sqrt{2}}\right)$

$\vec{a} = \frac{\mu}{3\sqrt{2}}\left(3\hat{i}+3\hat{j}-\hat{i}+\hat{j}-4\hat{k}\right)$

$= \frac{\mu}{3\sqrt{2}}\left(2\hat{i}+4\hat{j}-4\hat{k}\right)$

Compare withतुलना करने पर $\vec{a} = \alpha\hat{i}+2\hat{j}+\beta\hat{k}$

$\frac{4\mu}{3\sqrt{2}} = 2 \Rightarrow \mu = \frac{3\sqrt{2}}{2}$

$\vec{a} = \hat{i}+2\hat{j}-2\hat{k}$

$\vec{a}.\hat{k}+2 = 0$

$-2 + 2 = 0$

45. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three unit vectros such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. If $\lambda = \vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$ and $\vec{d} = \vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}$, then the ordered pair, $(\lambda, \vec{d})$ is equal to:

माना $\vec{a}, \vec{b}$ तथा $\vec{c}$ तीन मात्रक (unit) सदिश इस प्रकार हैं कि $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. यदि $\lambda = \vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$ तथा $\vec{d} = \vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}$, तो क्रमित युग्म $(\lambda, \vec{d})$ बराबर है–

(1) $\left(\frac{3}{2}, 3\vec{a}\times\vec{c}\right)$ (2) $\left(-\frac{3}{2}, 3\vec{c}\times\vec{b}\right)$ (3) $\left(\frac{3}{2}, 3\vec{b}\times\vec{c}\right)$ (4) $\left(-\frac{3}{2}, 3\vec{a}\times\vec{b}\right)$

Ans. (4) [JEE(Main) 2020, Online (07-01-20),P-2 (4, –1), 120]

Sol. $|\vec{a}+\vec{b}+\vec{c}|^2 = 0$

$3 + 2\,(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}) = 0$

$(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}) = \frac{-3}{2}$

$\Rightarrow \lambda = \frac{-3}{2}$

$\vec{d} = \vec{a}\times\vec{b} + \vec{b}\times(-\vec{a}-\vec{b}) + (-\vec{a}-\vec{b})\times\vec{a}$

$= \vec{a}\times\vec{b} + \vec{a}\times\vec{b} + \vec{a}\times\vec{b}$

$\vec{d} = 3\ (\vec{a}\times\vec{b})$

46. If the distance between the plane, 23x –10y –2z +48 = 0 and the plane containing the lines $\frac{x+1}{2} = \frac{y-3}{4} = \frac{z+1}{3}$ and $\frac{x+3}{2} = \frac{y+2}{6} = \frac{z-1}{\lambda}(\lambda \in R)$ is equal to $\frac{k}{\sqrt{633}}$, then k is equal to ___________

यदि समतल 23x –10y –2z +48 = 0तथा रेखाओं $\frac{x+1}{2} = \frac{y-3}{4} = \frac{z+1}{3}$ और $\frac{x+3}{2} = \frac{y+2}{6} = \frac{z-1}{\lambda}(\lambda \in R)$ को अंतर्विष्ट करने वाले समतल के बीच की दूरी $\frac{k}{\sqrt{633}}$ है, तो k बराबर है___________ ।

[JEE(Main) 2020, Online (09-01-20),P-2 (4, 0), 120]

Ans. 3

Sol. Lines must be intersecting

$\Rightarrow$ (2s – 1, 4s + 3, 3s –1) = (2t – 3, 6t – 2 , λt + 1)

2s–1 = 2t – 3, 4s + 3 = 6t–2 , 3s – 1 = λt + 1 $\quad \Rightarrow t = \frac{1}{2}, s = -\frac{1}{2}, \lambda = -7$

distance of plane contains given lines from given plane is same as distance between point (–3, –2,1) from given plane.

Required distance equal to $\frac{|-69+20-2+48|}{\sqrt{529+100+4}} = \frac{3}{\sqrt{633}} = \frac{k}{\sqrt{633}} \Rightarrow k = 3$

Sol. रेखाऐं प्रतिच्छेदी होगी

$\Rightarrow$ (2s – 1, 4s + 3, 3s –1) = (2t – 3, 6t – 2 , λt + 1)

2s–1 = 2t – 3, 4s + 3 = 6t–2 , 3s – 1 = λt + 1 $\quad \Rightarrow t = \frac{1}{2}, s = -\frac{1}{2}, \lambda = -7$

रेखाओं का समाहित करने वाले समतल की दिये हुये समतल से दूरी = बिंदु (–3, –2,1) से दिये गये समतल की दूरी

अभीष्ट दूरी = $\frac{|-69+20-2+48|}{\sqrt{529+100+4}} = \frac{3}{\sqrt{633}} = \frac{k}{\sqrt{633}} \Rightarrow k = 3$

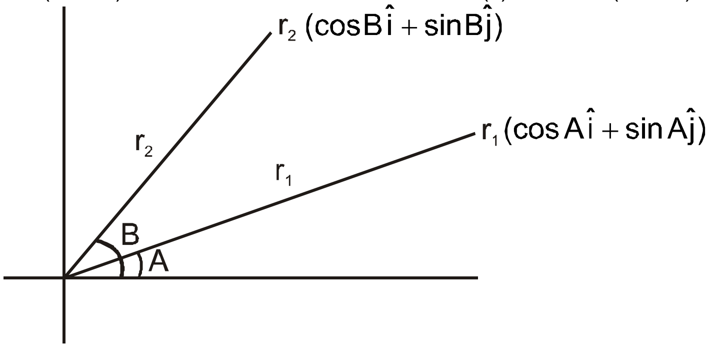
High Level Problems (HLP)

SUBJECTIVE QUESTIONS

विषयात्मक प्रश्न (SUBJECTIVE QUESTIONS)

1. Using Vectors prove that
सदिश विधि से दर्शाइये कि

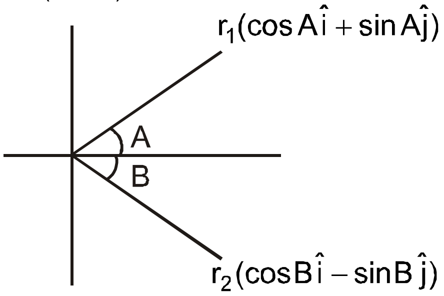
(i) $\cos(A - B) = \cos A \cos B + \sin A \sin B$ (ii) $\sin(A + B) = \sin A \cos B + \cos A \sin B$



Sol. **(i)**

$r_1 r_2 \cos(A - B) = r_1 r_2(\cos A\hat{i} + \sin A\hat{j}).(\cos B\hat{i} + \sin B\hat{j})$

$\cos(A - B) = \cos A \cos B + \sin A \sin B$



(ii)

$\vec{p} \times \vec{q} = r_1 r_2(\cos A\hat{i} + \sin A\hat{j}) \times (\cos B\hat{i} - \sin B\hat{j})$

$r_1 r_2 \sin(A + B)(-\hat{k}) = -\hat{k} r_1 r_2(\cos A \sin B + \sin A \cos B)$

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

2. Using vectors, prove that the altitudes of a triangle are concurrent.
सदिश विधि से सिद्व करो कि एक त्रिभुज के शीर्ष लम्ब संगामी होते है।

Sol. Let AD $\perp$ BC and BE $\perp$ AC. Let AD and BE meet at O. Join CO and produce it to meet AB at F.
Take O as the origin. Let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of A, B and C respectively.

$\therefore$ $\overrightarrow{BC} = \vec{c} - \vec{b}$ and $\overrightarrow{CA} = \vec{a} - \vec{c}$

Now $\overrightarrow{AD} \perp \overrightarrow{BC} \Rightarrow \overrightarrow{OA} \perp \overrightarrow{BC} \Rightarrow \vec{a}.(\vec{b} - \vec{c}) = 0$

and $\overrightarrow{BE} \perp \overrightarrow{CA} \Rightarrow \overrightarrow{OB} \perp \overrightarrow{CA} \Rightarrow \vec{b}.(\vec{a} - \vec{c}) = 0$

These imply

$\vec{a}.\vec{b} = \vec{a}.\vec{c}$ and $\vec{a}.\vec{b} = \vec{b}.\vec{c} \Rightarrow \vec{a}.\vec{c} = \vec{b}.\vec{c} \Rightarrow (\vec{b} - \vec{a}).\vec{c} = 0$

$\Rightarrow \overrightarrow{OC} \perp \overrightarrow{AB} \Rightarrow CF \perp AB$

$\Rightarrow$ Altitudes of a Δ are concurrent.

Hindi माना AD $\perp$ BC तथा BE $\perp$ AC माना AD तथा BE बिन्दु O पर मिलाता है। CO को मिलाने पर तथा इसको आगे बढाने पर यह AB को F पर मिलेगी।

माना O मूल बिन्दु है A, B, C के स्थिति सदिश क्रमशः $\vec{a}, \vec{b}, \vec{c}$ हैं।

$\therefore$ $\overrightarrow{BC} = \vec{c} - \vec{b}$ तथा $\overrightarrow{CA} = \vec{a} - \vec{c}$

अब $\overrightarrow{AD} \perp \overrightarrow{BC} \Rightarrow \overrightarrow{OA} \perp \overrightarrow{BC} \Rightarrow \vec{a}.(\vec{b}-\vec{c}) = 0$

तथा $\overrightarrow{BE} \perp \overrightarrow{CA} \Rightarrow \overrightarrow{OB} \perp \overrightarrow{CA} \Rightarrow \vec{b}.(\vec{a}-\vec{c}) = 0$

इसका मतलब है कि

$\vec{a}.\vec{b} = \vec{a}.\vec{c}$ तथा $\vec{a}.\vec{b} = \vec{b}.\vec{c} \Rightarrow \vec{a}.\vec{c} = \vec{b}.\vec{c} \Rightarrow (\vec{b}-\vec{a}).\vec{c} = 0$

$\Rightarrow \overrightarrow{OC} \perp \overrightarrow{AB} \Rightarrow CF \perp AB$

अतः त्रिभुज के शीर्ष लम्ब संगामी होगे।

3. Prove that the direction cosines of a line equally inclined to three mutually perpendicular lines having D.C.'s as ℓ_1, m_1, n_1 ; ℓ_2, m_2, n_2 ; ℓ_3, m_3, n_3 are $\frac{\ell_1+\ell_2+\ell_3}{\sqrt{3}}, \frac{m_1+m_2+m_3}{\sqrt{3}}, \frac{n_1+n_2+n_3}{\sqrt{3}}$.

सिद्ध कीजिए कि तीन परस्पर लम्बवत् रेखाओं जिनकी दिक्कोज्याऐं ℓ_1, m_1, n_1 ; ℓ_2, m_2, n_2 ; ℓ_3, m_3, n_3 है, से बराबर झुकी हुई रेखा की दिक्कोज्याऐं $\frac{\ell_1+\ell_2+\ell_3}{\sqrt{3}}, \frac{m_1+m_2+m_3}{\sqrt{3}}, \frac{n_1+n_2+n_3}{\sqrt{3}}$ है।

Sol. Let $\overrightarrow{OA} = \ell_1\hat{i}+m_1\hat{j}+n_1\hat{k}$, $\overrightarrow{OB} = \ell_2\hat{i}+m_2\hat{j}+n_2\hat{k}$ and $\overrightarrow{OC} = \ell_3\hat{i}+m_3\hat{j}+n_3\hat{k}$ be mutually perpendicular vectors. Let $\overrightarrow{OP} = \ell\hat{i}+m\hat{j}+n\hat{k}$ be equally inclined to $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$.

Then $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \Sigma\ell_1\hat{i}+\Sigma m_1\hat{j}+\Sigma n_1\hat{k}$

$$\left|\overrightarrow{OP}\right|^2 = (\Sigma\ell_1)^2+(\Sigma m_1)^2+(\Sigma n_1)^2 = 3 + 2\,\Sigma\,(\ell_1\ell_2 + m_1m_2 + n_1n_2) = 3$$

$$\therefore \quad \left|\overrightarrow{OP}\right| = \sqrt{3} \qquad \therefore \quad \widehat{OP} = \frac{\ell_1+\ell_2+\ell_3}{\sqrt{3}}\,\hat{i} + \frac{m_1+m_2+m_3}{\sqrt{3}}\,\hat{j} + \frac{n_1+n_2+n_3}{\sqrt{3}}\,\hat{k}$$

Hindi मानाकि $\overrightarrow{OA} = \ell_1\hat{i}+m_1\hat{j}+n_1\hat{k}$, $\overrightarrow{OB} = \ell_2\hat{i}+m_2\hat{j}+n_2\hat{k}$ और $\overrightarrow{OC} = \ell_3\hat{i}+m_3\hat{j}+n_3\hat{k}$ तीन परस्पर लम्बवत सदिश हैं। मानाकि $\overrightarrow{OP} = \ell\hat{i}+m\hat{j}+n\hat{k}$, $\overrightarrow{OA}$, $\overrightarrow{OB}$ तथा $\overrightarrow{OC}$ से समान कोण बनाता है, तो

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \Sigma\ell_1\hat{i}+\Sigma m_1\hat{j}+\Sigma n_1\hat{k}$

$$\left|\overrightarrow{OP}\right|^2 = (\Sigma\ell_1)^2+(\Sigma m_1)^2+(\Sigma n_1)^2 = 3 + 2\,\Sigma\,(\ell_1\ell_2 + m_1m_2 + n_1n_2) = 3$$

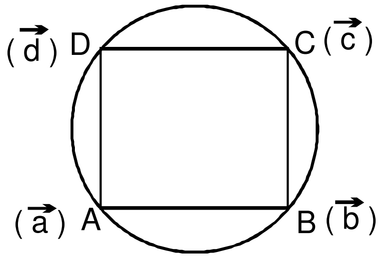
$$\therefore \quad \left|\overrightarrow{OP}\right| = \sqrt{3} \qquad \therefore \quad \widehat{OP} = \frac{\ell_1+\ell_2+\ell_3}{\sqrt{3}}\,\hat{i} + \frac{m_1+m_2+m_3}{\sqrt{3}}\,\hat{j} + \frac{n_1+n_2+n_3}{\sqrt{3}}\,\hat{k}$$

4. If $A(\vec{a})$, $B(\vec{b})$, $C(\vec{c})$, $D(\vec{d})$ are the position vector of cyclic quadrilateral then find the value of $\frac{|\vec{a}\times\vec{b}+\vec{b}\times\vec{d}+\vec{d}\times\vec{a}|}{[(\vec{b}-\vec{a}).(\vec{d}-\vec{a})]} + \frac{|\vec{b}\times\vec{c}+\vec{c}\times\vec{d}+\vec{d}\times\vec{b}|}{[(\vec{b}-\vec{c}).(\vec{d}-\vec{c})]}$. (It is given that no angle of cyclic quadrilateral ABCD is right angle)

यदि $A(\vec{a})$, $B(\vec{b})$, $C(\vec{c})$, $D(\vec{d})$ चक्रीय चतुर्भुज के स्थिति सदिश हो, तब $\frac{|\vec{a}\times\vec{b}+\vec{b}\times\vec{d}+\vec{d}\times\vec{a}|}{[(\vec{b}-\vec{a}).(\vec{d}-\vec{a})]} + \frac{|\vec{b}\times\vec{c}+\vec{c}\times\vec{d}+\vec{d}\times\vec{b}|}{[(\vec{b}-\vec{c}).(\vec{d}-\vec{c})]}$ का मान ज्ञात कीजिए। (यह दिया गया है कि चक्रीय चतुर्भुज ABCD का कोई भी कोण समकोण नहीं है।)

Ans. 0

Sol.



Now अब $|(\vec{b}-\vec{c}).(\vec{d}-\vec{a})|$ = AB.AD.cos A.............(1)

$|\vec{a}\times\vec{b}+\vec{b}\times\vec{d}+\vec{d}\times\vec{a}|$ = $|(\vec{b}-\vec{a})\times(\vec{d}-\vec{a})|$

$|(\vec{b}-\vec{c})\times(\vec{d}-\vec{a})|$ = AB.AD.sin A.............(2)

$(\vec{b}-\vec{c}).(\vec{d}-\vec{c})$ = BC.CD.cosC

$|(\vec{b}-\vec{c})\times(\vec{d}-\vec{c})|$ = BC.CD sinC

Now putting the values का मान रखने पर

tanA + tanC [$\therefore$ Now (A + C = 180)]

tanA + tan $(\pi - A)$

= 0

5. Prove that the volume of tetrahedron bounded by the planes,

$\vec{r}.(m\hat{j}+n\hat{k})=0$, $\vec{r}.(n\hat{k}+\ell\hat{i})=0$, $\vec{r}.(\ell\hat{i}+m\hat{j})=0$, $\vec{r}.(\ell\ \hat{i}+m\ \hat{j}+n\ \hat{k})$ = p is $\dfrac{2p^3}{3\ell mn}$.

सिद्ध कीजिए कि दिये गये समतलो $\vec{r}.(m\hat{j}+n\hat{k})=0$, $\vec{r}.(n\hat{k}+\ell\hat{i})=0$, $\vec{r}.(\ell\hat{i}+m\hat{j})=0$,

$\vec{r}.(\ell\ \hat{i}+m\ \hat{j}+n\ \hat{k})$ = p द्वारा घिरे समचतुष्फलक का आयतन $\dfrac{2p^3}{3\ell mn}$ है।

Sol. Vectors along the edges passing through the origin.

$(m\hat{j}+n\hat{k})\times(\ell\hat{i}+n\hat{k}) = mn\hat{i}+n\ell\hat{j}-\ell m\hat{k}$

$(n\hat{k}+\ell\hat{i})\times(\ell\hat{i}+m\hat{j}) = -mn\hat{i}+n\ell\hat{j}+\ell m\hat{k}$

$(\ell\hat{i}+m\hat{j})\times(m\hat{j}+n\hat{k}) = mn\hat{i}-n\ell\hat{j}+\ell m\hat{k}$

Let the equation of one edge be $\vec{r}=\lambda(mn\hat{i}+n\ell\hat{j}-\ell m\hat{k})$

For its point of intersection with the plane $\vec{r}.(\ell\hat{i}+m\hat{j}+n\hat{k})=p$, we have

$\lambda\ (mn\hat{i}+n\ell\hat{j}-\ell m\hat{k}).\ (\ell\hat{i}+m\hat{j}+n\hat{k}) = p \qquad \therefore \qquad \lambda = \dfrac{p}{\ell mn}$

$$\begin{vmatrix} mn & n\ell & -\ell m \\ -mn & n\ell & \ell m \\ mn & -n\ell & \ell m \end{vmatrix} = 4\ell^2m^2n^2$$

$\therefore$ volume of the tetrahedron = $\dfrac{1}{6}\dfrac{p^3}{(\ell mn)^3}.4\ell^2m^2n^2 = \dfrac{2}{3}\dfrac{p^3}{\ell mn}$

Hindi मूल बिन्दु से गुजरने वाले कोरों के अनुदिश सदिश

$(m\hat{j}+n\hat{k})\times(\ell\hat{i}+n\hat{k}) = mn\hat{i}+n\ell\hat{j}-\ell m\hat{k}$

$(n\hat{k}+\ell\hat{i})\times(\ell\hat{i}+m\hat{j}) = -mn\hat{i}+n\ell\hat{j}+\ell m\hat{k}$

$(\ell\hat{i}+m\hat{j})\times(m\hat{j}+n\hat{k}) = mn\hat{i}-n\ell\hat{j}+\ell m\hat{k}$

मानाकि एक कोर का समीकरण $\vec{r}=\lambda(mn\hat{i}+n\ell\hat{j}-\ell m\hat{k})$

इसका समतल $\vec{r}.(\ell\hat{i}+m\hat{j}+n\hat{k})=p$ के साथ प्रतिच्छेद बिन्दु के लिए

$\lambda(mn\hat{i}+n\ell\hat{j}-\ell m\hat{k})\ .\ (\ell\hat{i}+m\hat{j}+n\hat{k}) = p$

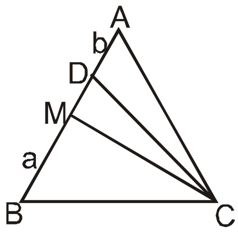
$\therefore \quad \lambda = \dfrac{p}{\ell mn} \qquad \because \begin{vmatrix} mn & n\ell & -\ell m \\ -mn & n\ell & \ell m \\ mn & -n\ell & \ell m \end{vmatrix} = 4\ell^2m^2n^2$

$\therefore$ चतुष्फलक का आयतन $= \dfrac{1}{6}\dfrac{p^3}{(\ell mn)^3} \cdot 4\ell^2m^2n^2 = \dfrac{2}{3}\dfrac{p^3}{\ell mn}$

6. In a Δ ABC, let M be the mid point of segment AB and let D be the foot of the bisector of $\angle$C. Then prove that $\dfrac{ar(\Delta CDM)}{ar(\Delta ABC)} = \dfrac{1}{2}\dfrac{a-b}{a+b} = \dfrac{1}{2}\tan\dfrac{A-B}{2}\cot\dfrac{A+B}{2}$.

किसी Δ ABC में भुजा AB का मध्य बिन्दु M है और D, $\angle$ C के अर्धक का पाद है। तब सिद्ध कीजिए $\dfrac{\Delta CDM \text{ का क्षे०}}{\Delta ABC \text{ का क्षे०}} = \dfrac{1}{2}\dfrac{a-b}{a+b} = \dfrac{1}{2}\tan\dfrac{A-B}{2}\cot\dfrac{A+B}{2}$

Sol.



$\overrightarrow{CD} = \dfrac{a\vec{a}+b\vec{b}}{a+b}$

$\overrightarrow{CM} = \dfrac{\vec{a}+\vec{b}}{2}$

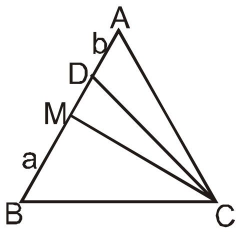
$\therefore \quad ar(\Delta\ CDM) = \dfrac{1}{2}|\overrightarrow{CD}\times\overrightarrow{CM}|$

$= \dfrac{1}{4} \times \dfrac{1}{(a+b)}\left|(a\vec{a}+b\vec{b})\times(\vec{a}+\vec{b})\right| = \dfrac{1}{4\ (a+b)}|(a-b)\ (\vec{a}\times\vec{b})|$

$= \dfrac{1}{2} \cdot \dfrac{a-b}{a+b} \cdot \dfrac{1}{2}|\vec{a}\times\vec{b}| = \dfrac{1}{2}\dfrac{a-b}{a+b}\ ar(\Delta ABC)$

$\therefore \quad \dfrac{ar(\Delta CDM)}{ar(\Delta ABC)} = \dfrac{1}{2}\dfrac{a-b}{a+b} = \dfrac{1}{2}\tan\dfrac{A-B}{2}\cot\dfrac{A+B}{2}$

Hindi.



$\overrightarrow{CD} = \dfrac{a\vec{a}+b\vec{b}}{a+b}$

$\overrightarrow{CM} = \dfrac{\vec{a}+\vec{b}}{2}$

$\therefore \quad \Delta$ CDM का क्षेत्रफल $= \dfrac{1}{2}|\overrightarrow{CD}\times\overrightarrow{CM}|$

$= \dfrac{1}{4} \times \dfrac{1}{(a+b)}\left|(a\vec{a}+b\vec{b})\times(\vec{a}+\vec{b})\right| = \dfrac{1}{4\ (a+b)}|(a-b)\ (\vec{a}\times\vec{b})|$

$$= \frac{1}{2} \cdot \frac{a-b}{a+b} \cdot \frac{1}{2}|\vec{a}\times\vec{b}| = \frac{1}{2}\,\frac{a-b}{a+b}\ \text{ar}(\Delta ABC)$$

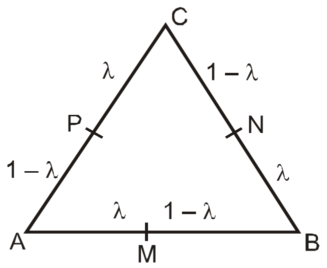
$$\therefore \quad \frac{\text{ar}(\Delta CDM)}{\text{ar}(\Delta ABC)} = \frac{1}{2}\,\frac{a-b}{a+b} = \frac{1}{2}\tan\frac{A-B}{2}\cot\frac{A+B}{2}$$

7. Let ABC be a triangle.Points M, N and P are taken on the sides AB, BC and CA respectively such that $\frac{AM}{AB} = \frac{BN}{BC} = \frac{CP}{CA} = \lambda$. Prove that the vectors $\overrightarrow{AN}$, $\overrightarrow{BP}$ and $\overrightarrow{CM}$ form a triangle. Also find λ for which the area of the triangle formed by these vectors is the least.

Ans. $\lambda = \frac{1}{2}$

माना ABC एक त्रिभुज है। भुजा AB, BC और CA पर क्रमशः बिन्दु M, N और P इस प्रकार से है कि $\frac{AM}{AB} = \frac{BN}{BC} = \frac{CP}{CA} = \lambda$ सिद्ध कीजिए कि सदिश $\overrightarrow{AN}$, $\overrightarrow{BP}$ तथा $\overrightarrow{CM}$ एक त्रिभुज बनाते है। λ का वह मान ज्ञात कीजिए जिसके लिए इन सदिशों द्वारा निर्मित त्रिभुज का क्षेत्रफल न्यूनतम हो।

Sol.



Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be the position vectors of A, B, C respectively.

$\overrightarrow{OM} = \lambda\vec{b} + (1-\lambda)\vec{a}$, $\overrightarrow{ON} = \lambda\vec{c} + (1-\lambda)\vec{b}$, $\overrightarrow{OP} = \lambda\vec{a} + (1-\lambda)\vec{c}$

$\therefore \quad \overrightarrow{AN} = \lambda\vec{c} + (1-\lambda)\vec{b} - \vec{a}$

$\overrightarrow{BP} = \lambda\vec{a} + (1-\lambda)\vec{c} - \vec{b}$ and $\overrightarrow{CM} = \lambda\vec{b} + (1-\lambda)\vec{a} - \vec{c}$

Clearly $\overrightarrow{AN} + \overrightarrow{BP} + \overrightarrow{CM} = \vec{0}$

$\therefore \quad \overrightarrow{AN}$, $\overrightarrow{BP}$, $\overrightarrow{CM}$ form a triangle

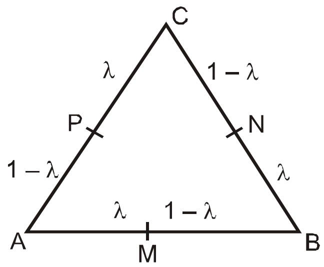
Area of triangle formed by the vectors $\overrightarrow{AN}$, $\overrightarrow{BP}$, $\overrightarrow{CM}$ is

$$= \frac{1}{2}|(\lambda\vec{c} + (1-\lambda)\vec{b} - \vec{a}) \times (\lambda\vec{a} + (1-\lambda)\vec{c} - \vec{b})| = \frac{1}{2}|\lambda^2 - \lambda + 1|\ (\Delta\, ABC)$$

is least if $|\lambda^2 - \lambda + 1|$ is least

Which is when $\lambda = \frac{1}{2}$

Hindi.



मानाकि A, B, C के स्थिति सदिश क्रमशः $\vec{a}$, $\vec{b}$ व $\vec{c}$ है।

$\overrightarrow{OM} = \lambda\vec{b} + (1-\lambda)\vec{a}$, $\overrightarrow{ON} = \lambda\vec{c} + (1-\lambda)\vec{b}$, $\overrightarrow{OP} = \lambda\vec{a} + (1-\lambda)\vec{c}$

$\therefore \quad \overrightarrow{AN} = \lambda\vec{c} + (1-\lambda)\vec{b} - \vec{a}$

$\overrightarrow{BP} = \lambda\vec{a} + (1-\lambda)\vec{c} - \vec{b}$ तथा $\overrightarrow{CM} = \lambda\vec{b} + (1-\lambda)\vec{a} - \vec{c}$

स्पष्टतया $\overrightarrow{AN} + \overrightarrow{BP} + \overrightarrow{CM} = \vec{0}$

$\therefore$ $\overrightarrow{AN}$, $\overrightarrow{BP}$, $\overrightarrow{CM}$ एक त्रिभुज बनाते हैं।

सदिशों $\overrightarrow{AN}$, $\overrightarrow{BP}$, $\overrightarrow{CM}$ से निर्मित त्रिभुज का क्षेत्रफल है।

$$= \frac{1}{2}|(\lambda\vec{c} + (1-\lambda)\ \vec{b} - \vec{a}) \times (\lambda\vec{a} + (1-\lambda)\ \ \vec{c} - \vec{b})| = \frac{1}{2}|\lambda^2 - \lambda + 1|\ (\Delta\,ABC)$$

यह न्यूनतम होगा यदि $|\lambda^2 - \lambda + 1|$ न्यूनतम हो।

जो कि सम्भव है यदि $\lambda = \frac{1}{2}$ हो।

8. In any triangle, show that the perpendicular bisectors of the sides are concurrent.
किसी त्रिभुज में सिद्ध कीजिए कि भुजाओं के लम्ब अर्द्धक संगामी है।

Sol. Let ABC be the triangle and D, E and F are respectively middle points of sides BC, CA and AB. Let the perpendicular bisectors of BC and CA meet at O. Join OF. We are required to $\vec{a}$ prove that OF is $\perp$ to AB. Let the position vectors of A, B, C with O as origin of reference be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively.

$\therefore$ $\overrightarrow{OD} = \frac{1}{2}(\vec{b} + \vec{c}), \overrightarrow{OE} = \frac{1}{2}(\vec{c} + \vec{a})$ and $\overrightarrow{OF} = \frac{1}{2}(\vec{a} + \vec{b})$

Also $\overrightarrow{BC} = \vec{c} - \vec{b}$, $\overrightarrow{CA} = \vec{a} - \vec{c}$ and $\overrightarrow{AB} = \vec{b} - \vec{a}$

Since OD $\perp$ BC

$\Rightarrow \frac{1}{2}(\vec{b} + \vec{c}).(\vec{c} - \vec{b}) = 0 \quad \Rightarrow b^2 = c^2$ ....(i)

Similarly OE $\perp$ CA $\quad \Rightarrow \frac{1}{2}(\vec{c} + \vec{a}).(\vec{a} - \vec{c}) = 0 \quad \Rightarrow a^2 = c^2$ ....(ii)

from (i) and (ii) we have $a^2 - b^2 = 0$

$\Rightarrow (\vec{a} + \vec{b}).(\vec{b} - \vec{a}) = 0 \quad \Rightarrow \frac{1}{2}(\vec{b} + \vec{a}).(\vec{b} - \vec{a}) = 0 \quad \Rightarrow \overrightarrow{OF} \perp \overrightarrow{AB}$.

Sol. माना ABC एक त्रिभुज है तथा D, E तथा F क्रमशः भुजाओं BC, CA तथा AB के मध्य बिन्दु है। माना BC और CA का लम्ब अर्द्धक बिन्दु O पर मिलते है। OF को मिलाइये। हमें $\vec{a}$ की आवश्यकता है तब सिद्ध करना है कि OF $\perp$ AB माना A, B, C के O के सापेक्ष स्थिति सदिश क्रमशः $\vec{a}$, $\vec{b}$ और $\vec{c}$ है।

$\therefore$ $\overrightarrow{OD} = \frac{1}{2}(\vec{b} + \vec{c}), \overrightarrow{OE} = \frac{1}{2}(\vec{c} + \vec{a})$ तथा $\overrightarrow{OF} = \frac{1}{2}(\vec{a} + \vec{b})$

तथा $\overrightarrow{BC} = \vec{c} - \vec{b}$, $\overrightarrow{CA} = \vec{a} - \vec{c}$ और $\overrightarrow{AB} = \vec{b} - \vec{a}$

चूंकि OD $\perp$ BC

$\Rightarrow \frac{1}{2}(\vec{b} + \vec{c}).(\vec{c} - \vec{b}) = 0 \quad \Rightarrow b^2 = c^2$ ....(i)

इसीप्रकार OE $\perp$ CA $\quad \Rightarrow \frac{1}{2}(\vec{c} + \vec{a}).(\vec{a} - \vec{c}) = 0 \quad \Rightarrow a^2 = c^2$ ....(ii)

समीकरण (i) और (ii) से $a^2 - b^2 = 0$

$\Rightarrow (\vec{a} + \vec{b}).(\vec{b} - \vec{a}) = 0 \quad \Rightarrow \frac{1}{2}(\vec{b} + \vec{a}).(\vec{b} - \vec{a}) = 0 \quad \Rightarrow \overrightarrow{OF} \perp \overrightarrow{AB}$.

9. Let ABC be an acute-angled triangle AD be the bisector of $\angle$BAC with D on BC and BE be the altitude from B on AC. Show that $\angle$CED > 45º.
किसी न्यूनकोण त्रिभुज ABC में माना $\angle$BAC का अर्धक AD इस प्रकार है कि बिन्दु D भुजा BC पर स्थित है और शीर्ष B से भुजा AC पर डाला गया लम्ब BE है। प्रदर्शित कीजिए कि $\angle$CED > 45º.

Sol. Let AB = c′ and AC = b′

p.v. of D is $\dfrac{c'\vec{c}+b'\vec{b}}{c'+b'}$

Also $(\vec{a}-\vec{c}).\vec{b} = 0$

Now $\cos\angle CED = \dfrac{\dfrac{c'\ \vec{c}+b'\ \vec{b}}{c'+b'}.\ \vec{c}}{\left|\dfrac{c'\ \vec{c}+b'\ \vec{b}}{c'+b'}\right|.\ |\ \vec{c}\ |}$

$\Rightarrow \quad \cos\theta = \dfrac{c'\ |\ \vec{c}\ |^2}{|c'\ \vec{c}+b'\ \vec{b}|\ \ |\ \vec{c}\ |} = \dfrac{c'\ |\ \vec{c}\ |}{\sqrt{(\ c'\ )^2|\vec{c}|^2 + (\ b'\ )^2|\ \vec{b}|^2}}$

$\Rightarrow \quad (b')^2|\ \vec{b}|^2\cos^2\theta = (c')^2\ |\vec{c}|^2\sin^2\theta \quad \Rightarrow \quad \dfrac{\cos^2\theta}{\sin^2\theta} = \dfrac{\sin^2 C}{\sin^2 B} \times \dfrac{1}{\tan^2 C}$

$\Rightarrow \quad \cot^2\theta = \dfrac{\cos^2 C}{\sin^2 B}$ ... (i)

Since A,B,C are all acute.

Hence to prove that $\dfrac{\cos^2 C}{\sin^2 B} < 1$

or $\cos C < \sin B$ Now since $B + C > \dfrac{\pi}{2} \Rightarrow B > \dfrac{\pi}{2} - C$

or $\sin B > \cos C \Rightarrow$ from equation (i) $\cot\theta < 1 \Rightarrow \theta > 45°$

Hence Proved.

Hindi. माना $AB = c'$ और $AC = b'$

बिन्दु D का स्थिति सदिश $\dfrac{c'\vec{c}+b'\vec{b}}{c'+b'}$

तथा $(\vec{a}-\vec{c})\ .\ \vec{b} = 0$

$\because \quad \cos\angle CED = \dfrac{\dfrac{c'\ \vec{c}+b'\ \vec{b}}{c'+b'}.\ \vec{c}}{\left|\dfrac{c'\ \vec{c}+b'\ \vec{b}}{c'+b'}\right|.\ |\ \vec{c}\ |}$

$\Rightarrow \quad \cos\theta = \dfrac{c'\ |\ \vec{c}\ |^2}{|c'\ \vec{c}+b'\ \vec{b}|\ \ |\ \vec{c}\ |} = \dfrac{c'\ |\ \vec{c}\ |}{\sqrt{(\ c'\ )^2|\vec{c}|^2 + (\ b'\ )^2|\ \vec{b}|^2}}$

$\Rightarrow \quad (b')^2\ |\ \vec{b}|^2\ \cos^2\theta = (c')^2|\vec{c}|^2\ \sin^2\theta$

$\Rightarrow \quad \dfrac{\cos^2\theta}{\sin^2\theta} = \dfrac{\sin^2 C}{\sin^2 B} \times \dfrac{1}{\tan^2 C} \quad \Rightarrow \quad \cot^2\theta = \dfrac{\cos^2 C}{\sin^2 B}$ ... (i)

चूँकि कोण A, B, C सभी न्यूनकोण है।

अतः यह सिद्ध करने के लिये कि $\dfrac{\cos^2 C}{\sin^2 B} < 1$ या $\cos C < \sin B$

$\because \quad B + C > \dfrac{\pi}{2} \Rightarrow B > \dfrac{\pi}{2} - C$

या $\sin B > \cos C \Rightarrow$ समीकरण (i) से $\cot\theta < 1 \Rightarrow \theta > 45°$

अतः सिद्ध हुआ।

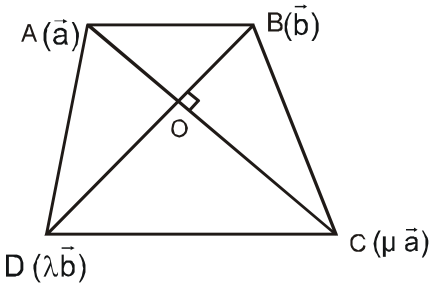
10. In a quadrilateral ABCD, it is given that AB || CD and the diagonals AC and BD are perpendicular to each other. Show that

(a) $AD.\ BC \geq AB.CD$ (b) $AD + BC \geq AB + CD$

किसी चतुर्भुज ABCD में भुजाएँ AB और CD परस्पर समान्तर है और विकर्ण AC और BD परस्पर लम्बवत् है।
प्रदर्शित कीजिए कि

(a) $AD.\ BC \geq AB.CD$ (b) $AD + BC \geq AB + CD$

Sol.



$\vec{a}\,.\,\vec{b} = 0$

Also $\mu\vec{a} - \lambda\vec{b} \,||\, \vec{a} - \vec{b} \Rightarrow \mu = \lambda.$

(a) $|\lambda\vec{b} - \vec{a}|\,|\mu\vec{a} - \vec{b}| \geq |(\vec{b} - \vec{a})|\,|(\lambda\vec{b} - \mu\vec{a})|$

$\Rightarrow$ $(\lambda^2 b^2 + a^2)(\mu^2a^2 + b^2) \geq (b^2 + a^2)(\lambda^2 b^2 + \mu^2a^2)$

$\Rightarrow$ $\lambda^2 \mu^2 a^2 b^2 + a^2b^2 \geq \lambda^2 a^2 b^2 + \mu^2 b^2 a^2$

$\Rightarrow$ $(\lambda^2 - 1)(\mu^2 - 1) \geq 0$

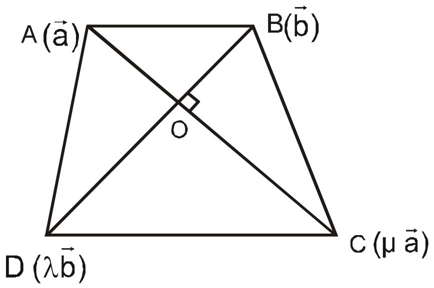
$\Rightarrow$ $(\lambda^2 - 1)^2 \geq 0$ which is true.

(b) $(AD + BC)^2 \geq (AB + CD)^2$

$\Rightarrow$ $AD^2 + OD^2 + OC^2 + OB^2 + 2AD.BC \geq OA^2 + OB^2 + OD^2 + OC^2 + 2AB.\ CD$

which follows from (a).

Hindi.



$\vec{a}\,.\,\vec{b} = 0$

Also $\mu\vec{a} - \lambda\vec{b} \,||\, \vec{a} - \vec{b} \Rightarrow \mu = \lambda.$

(a) $|\lambda\vec{b} - \vec{a}|\,|\mu\vec{a} - \vec{b}| \geq |(\vec{b} - \vec{a})|\,|(\lambda\vec{b} - \mu\vec{a})|$

$\Rightarrow$ $(\lambda^2 b^2 + a^2)(\mu^2a^2 + b^2) \geq (b^2 + a^2)(\lambda^2 b^2 + \mu^2a^2)$

$\Rightarrow$ $\lambda^2 \mu^2 a^2 b^2 + a^2b^2 \geq \lambda^2 a^2 b^2 + \mu^2 b^2 a^2$

$\Rightarrow$ $(\lambda^2 - 1)(\mu^2 - 1) \geq 0$

$\Rightarrow$ $(\lambda^2 b^2 + a^2)(\mu^2a^2 + b^2) \geq (b^2 + a^2)(\lambda^2 b^2 + \mu^2a^2)$

$\Rightarrow$ $\lambda^2 \mu^2 a^2 b^2 + a^2b^2 \geq \lambda^2 a^2 b^2 + \mu^2 b^2 a^2$

$\Rightarrow$ $(\lambda^2 - 1)(\mu^2 - 1) \geq 0$

$\Rightarrow$ $(\lambda^2 - 1)^2 \geq 0$ जो सत्य है।

(b) $(AD + BC)^2 \geq (AB + CD)^2$

$\Rightarrow$ $AD^2 + OD^2 + OC^2 + OB^2 + 2AD.BC \geq OA^2 + OB^2 + OD^2 + OC^2 + 2AB.\ CD$

11. A, B, C, D are four points in space. using vector methods, prove that
$AC^2 + BD^2 + AD^2 + BC^2 \geq AB^2 + CD^2$ what is the implication of the sign of equality.
समष्टि में A, B, C, D चार बिन्दु है सदिश विधि से सिद्ध कीजिए कि $AC^2 + BD^2 + AD^2 + BC^2 \geq AB^2 + CD^2$ असमिका के चिन्ह का अर्थ समझाइये।

Sol. Let the position vector of A, B, C, D be $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ respectively then

$AC^2+BD^2 + AD^2 + BC^2 = (\vec{c}-\vec{a}).(\vec{c}-\vec{a}) + (\vec{d}-\vec{b}).(\vec{d}-\vec{b}) + (\vec{d}-\vec{a}).(\vec{d}-\vec{a}) + (\vec{c}-\vec{b}).(\vec{c}-\vec{b}) = |\vec{c}|^2$

$+|\vec{a}|^2 - 2\vec{a}.\vec{c} + |\vec{d}|^2 + |\vec{b}|^2 - 2\vec{d}.\vec{b} + |\vec{d}|^2 + |\vec{a}|^2 - 2\vec{a}.\vec{d} + |\vec{c}|^2 + |\vec{b}|^2 - 2\vec{b}.\vec{c}$

$= |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}.\vec{b} + |\vec{c}|^2 + |\vec{d}|^2 - 2\vec{c}.\vec{d} + |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + |\vec{d}|^2$

$+ 2\vec{a}.\vec{b} + 2\vec{c}.\vec{d} - 2\vec{a}.\vec{c} - 2\vec{b}.\vec{d} - 2\vec{a}.\vec{d} - 2\vec{b}.\vec{c}$

$= (\vec{a}-\vec{b}).(\vec{a}-\vec{b}) + (\vec{c}-\vec{d}).(\vec{c}-\vec{d}) + (\vec{a}+\vec{b}-\vec{c}-\vec{d})^2$

$= AB^2 + CD^2 + (\vec{a}+\vec{b}-\vec{c}-\vec{d}).(\vec{a}+\vec{b}-\vec{c}-\vec{d}) \ge AB^2 + CD^2$

$\Rightarrow$ $AC^2 + BD^2 + AD^2 + BC^2 \ge AB^2 + CD^2$

for the sign of equality to hold, $\vec{a}+\vec{b}-\vec{c}-\vec{d} = 0$ or $\vec{a}-\vec{c} = \vec{d}-\vec{b}$

$\Rightarrow$ $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are collinear, the four points A, B, C, D are collinear

Hindi. माना A, B, C, D के स्थिति सदिश $\vec{a}$, $\vec{b}$, $\vec{c}$ और $\vec{d}$ है तब

$AC^2+BD^2 + AD^2 + BC^2 = (\vec{c}-\vec{a}).(\vec{c}-\vec{a}) + (\vec{d}-\vec{b}).(\vec{d}-\vec{b}) + (\vec{d}-\vec{a}).(\vec{d}-\vec{a}) + (\vec{c}-\vec{b}).(\vec{c}-\vec{b}) = |\vec{c}|^2$

$+|\vec{a}|^2 - 2\vec{a}.\vec{c} + |\vec{d}|^2 + |\vec{b}|^2 - 2\vec{d}.\vec{b} + |\vec{d}|^2 + |\vec{a}|^2 - 2\vec{a}.\vec{d} + |\vec{c}|^2 + |\vec{b}|^2 - 2\vec{b}.\vec{c}$

$= |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}.\vec{b} + |\vec{c}|^2 + |\vec{d}|^2 - 2\vec{c}.\vec{d} + |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + |\vec{d}|^2$

$+ 2\vec{a}.\vec{b} + 2\vec{c}.\vec{d} - 2\vec{a}.\vec{c} - 2\vec{b}.\vec{d} - 2\vec{a}.\vec{d} - 2\vec{b}.\vec{c}$

$= (\vec{a}-\vec{b}).(\vec{a}-\vec{b}) + (\vec{c}-\vec{d}).(\vec{c}-\vec{d}) + (\vec{a}+\vec{b}-\vec{c}-\vec{d})^2$

$= AB^2 + CD^2 + (\vec{a}+\vec{b}-\vec{c}-\vec{d}).(\vec{a}+\vec{b}-\vec{c}-\vec{d}) \ge AB^2 + CD^2$

$\Rightarrow$ $AC^2 + BD^2 + AD^2 + BC^2 \ge AB^2 + CD^2$

समिका के चिन्ह के लिए, $\vec{a}+\vec{b}-\vec{c}-\vec{d} = 0$ या $\vec{a}-\vec{c} = \vec{d}-\vec{b}$

$\Rightarrow$ $\overrightarrow{AC}$ और $\overrightarrow{BD}$ संरेखीय है तब चार बिन्दु A, B, C, D संरेखीय है।

12. The direction cosines of a variable line in two near by positions are l, m, n; $l + \delta l$, m + δm, n + δn. Show that the small angle δθ between the two position is given by $(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$.
दो निकटतम स्थितियों में एक चर रेखा की दिक्कोज्याएं l, m, n; $l + \delta l$, m + δm, n + δn हैं। प्रदर्शित कीजिए कि इन दो स्थितियों के मध्य लघु कोण δθ को $(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$ के द्वारा दिया जाता है।

Sol. As (l, m, n) and ($l + \delta l$, m + δm, n + δn) are the direction cosines of lines so we have

$l^2 + m^2 + n^2 = 1$ ... (1)

and $(l + \delta l)^2 + (m + \delta m)^2 + (n + \delta n)^2 = 1$ .... (2)

Subtracting (1) from (2) we have

$2(l\delta l + m\delta m + n\delta n) + [(\delta l)^2 + (\delta m)^2 + (\delta n)^2] = 0$

or $2\Sigma\,(l\delta l) = -\Sigma(\delta l)^2$ ... (3)

Also as δθ is the angle between these lines so we have

$\cos\delta\theta = l\,(l + \delta l) + m\,(m + \delta m) + n(n + \delta n)$

$= (l^2 + m^2 + n^2) + (l\delta l + m\delta m + n\delta n)$

$= 1 - \frac{1}{2}[\Sigma(\delta\ell)^2]$, from (1) and (3) or $\frac{1}{2}\left[\Sigma(\delta\ell)^2\right] = 1 - \cos\delta\theta = 2\sin^2\left(\frac{1}{2}\delta\theta\right) = 2\left(\frac{1}{2}\delta\theta\right)^2 = \frac{(\delta\theta)^2}{2}$

$\because$ $\sin\left(\frac{1}{2}\delta\theta\right) = \frac{1}{2}\delta\theta$ or $(\delta l)^2 + (\delta m)^2 + (\delta n)^2 = (\delta\theta)^2$

Hindi. रेखाओं की दिक्कोज्याएँ (l, m, n) तथा ($l + \delta l$, m + δm, n + δn) हैं। अतः

$l^2 + m^2 + n^2 = 1$ ... (1)

तथा $(l + \delta l)^2 + (m + \delta m)^2 + (n + \delta n)^2 = 1$ .... (2)

समीकरण (2) में से समीकरण (1) को घटाने पर

$2(l\delta l + m\delta m + n\delta n) + [(\delta l)^2 + (\delta m)^2 + (\delta n)^2] = 0$

या $2\Sigma\,(l\delta l) = -\Sigma(\delta l)^2$ .... (3)

इन दो रेखाओं के मध्य कोण δθ है। अतः

$\cos\delta\theta = l\,(l + \delta l) + m\,(m + \delta m) + n\,(n + \delta n)$
$= (l^2 + m^2 + n^2) + (l\delta l + m\delta m + n\delta n)$
$= 1 - \frac{1}{2}[\Sigma(\delta\ell)^2]$ समीकरण (1) तथा (3) से

या $\quad \frac{1}{2}\left[\Sigma(\delta\ell)^2\right] = 1 - \cos\delta\theta = 2\sin^2\left(\frac{1}{2}\delta\theta\right) = 2\left(\frac{1}{2}\delta\theta\right)^2 = \frac{(\delta\theta)^2}{2}$

$\because \quad \sin\left(\frac{1}{2}\delta\theta\right) = \frac{1}{2}\delta\theta$ या $(\delta l)^2 + (\delta m)^2 + (\delta n)^2 = (\delta\theta)^2$ जो कि (a) का अनुसंरण करते हुए है।

13. In a ΔABC, prove that distance between centroid and circumcentre is $\sqrt{R^2 - \left(\frac{a^2+b^2+c^2}{9}\right)}$

where R is the circumradius and a, b, c denotes the sides of ΔABC.

ΔABC में सिद्ध कीजिए कि केन्द्रक और परिकेन्द्र के मध्य दूरी $\sqrt{R^2 - \left(\frac{a^2+b^2+c^2}{9}\right)}$

जहां R परित्रिज्या है तथा ΔABC की भुजाएं a, b, c है।

Sol. Let the circum centre be $(\vec{O})$

माना $(\vec{O})$ परिकेन्द्र है।

$G = \left(\frac{\vec{a}+\vec{b}+\vec{c}}{3}\right)$

$\overrightarrow{OG} = \frac{1}{3}|\vec{a}+\vec{b}+\vec{c}|$

$= \frac{1}{3}\sqrt{|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a})}$

$|\vec{a}|^2 = |\vec{b}|^2 = |\vec{c}|^2 = R$

$\frac{1}{3}\sqrt{3R^2 + 2R^2(\cos 2A + \cos 2B + \cos 2C)}$

$\frac{1}{3}\sqrt{3R^2 + 2R^2(1 - 2\sin^2 A + 1 - 2\sin^2 B + 1 - 2\sin^2 C}$

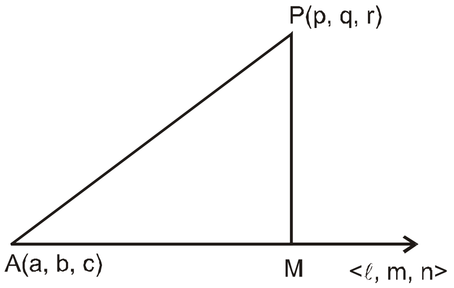
$\frac{1}{3}\sqrt{9R^2 - (a^2 + b^2 + c^2)}$

$\sqrt{R^2 - \left(\frac{a^2+b^2+c^2}{9}\right)}$

14. Prove that the square of the perpendicular distance of a point P (p, q, r) from a line through A(a, b, c) and whose direction cosines are ℓ, m, n is $\Sigma\{(q - b)\,n - (r - c)\,m\}^2$.

सिद्ध कीजिए कि बिन्दु A (a,b,c) से गुजरने वाली रेखा जिसकी दिककोज्याएँ ℓ, m, n हैं की बिन्दु P(p,q,r) से लम्बवत दूरी का वर्ग $\Sigma\{(q - b)\,n - (r - c)\,m\}^2$ है

Sol.

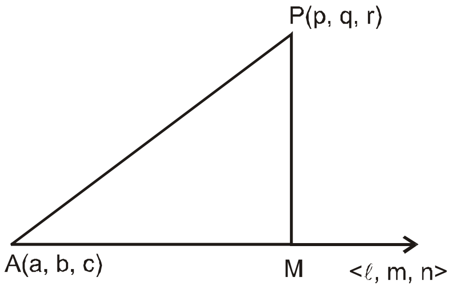


Consider $\overrightarrow{AP} \times (\ell\hat{i} + m\hat{j} + n\hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p-a & q-b & r-c \\ \ell & m & n \end{vmatrix}$

$= \Sigma\,(n\,(q-b) - m\,(r-c))\,\hat{i}$

$\therefore \quad MP^2 = \left|\overrightarrow{AP} \times (\ell\hat{i} + m\hat{j} + n\hat{k})\right|^2 = \Sigma\,\{n\,(q-b) - m\,(r-c)\}^2$

Hindi.



मानाकि $\overrightarrow{AP} \times (\ell\hat{i} + m\hat{j} + n\hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p-a & q-b & r-c \\ \ell & m & n \end{vmatrix}$

$= \Sigma\,(n\,(q-b) - m\,(r-c))\,\hat{i}$

$\therefore \quad MP^2 = \left|\overrightarrow{AP} \times (\ell\hat{i} + m\hat{j} + n\hat{k})\right|^2 = \Sigma\,\{(n\,(q-b) - m\,(r-c)\}^2$

15. **(i)** Let ℓ_1 & ℓ_2 be two skew lines. If P, Q are two distinct points on ℓ_1 and R, S are two distinct points on ℓ_2, then prove that PR can not be parallel to QS.

(ii) A line with direction cosines proportional to (2, 7 – 5) is drawn to intersect the lines $\frac{x-5}{3} = \frac{y-7}{-1} = \frac{z+2}{1}$ and $\frac{x+3}{-3} = \frac{y-3}{2} = \frac{z-6}{4}$. Find the coordinate of the points of intersection and the length intercepted on it. Also find the equation of intersecting straight line.

(i) मानाकि ℓ_1 & ℓ_2 दो विषमतलीय रेखाएँ (skew lines) हैं। यदि ℓ_1 पर स्थित दो भिन्न बिन्दु P, Q हैं तथा ℓ_2 पर स्थित दो भिन्न बिन्दु R, S है, तो सिद्ध कीजिए PR, QS के समान्तर नहीं हो सकती है।

(ii) एक रेखा जिसकी दिक्‌कोज्याएँ (2, 7 – 5) के समानुपाती हैं, रेखाओं $\frac{x-5}{3} = \frac{y-7}{-1} = \frac{z+2}{1}$ एवं $\frac{x+3}{-3} = \frac{y-3}{2} = \frac{z-6}{4}$ को प्रतिच्छेद करती है। प्रतिच्छेदन बिन्दुओं के निर्देशांक तथा इस पर बने अन्तःखण्डित भाग की लम्बाई ज्ञात कीजिए। साथ ही प्रतिच्छेदन सरल रेखा की समीकरण भी ज्ञात कीजिए।

Ans. (2, 8, – 3) & (0, 1, 2) ; $\sqrt{78}$; $\frac{x-2}{2} = \frac{y-8}{7} = \frac{z+3}{-5}$

Sol. (i) Let equation of the line ℓ_1 be $\vec{r} = \vec{a}+\lambda\vec{b}$ and equation of the line ℓ_2 be $\vec{r} = \vec{c}+\mu\vec{d}$, where $\vec{a}-\vec{c}$, $\vec{b}$ and $\vec{d}$ are non-coplanar.

Let the position vectors of points P and Q be $\vec{a}+\lambda_1\vec{b}$ and $\vec{a}+\lambda_2\vec{b}$ respectively.

Let the position vectors of points R and S be $\vec{c}+\mu_1\vec{d}$ and $\vec{c}+\mu_2\vec{d}$ respectively.

Then the lines PR and QS are parallel if and only if $\vec{c}-\vec{a}+\mu_1\vec{d}-\lambda_1\vec{b}=k\,(\vec{c}-\vec{a}+\mu_2\vec{d}-\lambda_2\vec{b})$

i.e. $(1-k)(\vec{c}-\vec{a}) + (\mu_1-k\mu_2)\vec{d} - (\lambda_1-k\lambda_2)\vec{b} = \vec{o}$

$\therefore$ $1-k=0,\ \mu_1-k\mu_2 = 0,\ \lambda_1-k\lambda_2=0$

i.e. $\mu_1=\mu_2$ and $\lambda_1=\lambda_2$ which is not possible $\therefore$ PR can not be parallel to QS.

(ii) The given equation

$$\frac{x-5}{3}=\frac{y-7}{-1}=\frac{z+2}{1} \quad \text{... (i)} \qquad \text{and} \qquad \frac{x+3}{-3}=\frac{y-3}{2}=\frac{z-6}{4} \quad \text{... (ii)}$$

any point P on (i) is $(3r_1+5, -r_1+7, r_1-2)$ and any point Q on (ii) is $(-3r_2-3, 2r_2+3, 4r_2+6)$

the direction ratios of PQ are

$(3r_1+3r_2+8, -r_1-2r_2+4, r_1-4r_2-8)$ ... (iii)

suppose the line with d.r's 2, 7, – 5 will be proportional to the d.r.'s given by (iii)

$$\therefore \quad \frac{3r_1+3r_2+8}{2}=\frac{-\ r_1-2r_2+4}{7}=\frac{r_1-4r_2-8}{-5} \quad \text{... (iv)}$$

Solving (iv), we get $r_1=r_2=-1$

So point of intersection are P(2, 8, – 3) and Q(0, 1, 2)

and intercepted length $= PQ = \sqrt{(2-0)^2+(8-1)^2+(-\ 3-2)^2} = \sqrt{78}$

and equation of PQ is $\dfrac{x-2}{2}=\dfrac{y-8}{7}=\dfrac{z+3}{-5}$

Hindi (i) माना कि रेखा ℓ_1 का समीकरण $\vec{r} = \vec{a}+\lambda\vec{b}$ है तथा रेखा ℓ_2 का समीकरण $\vec{r} = \vec{c}+\mu\vec{d}$ है। जहाँ $\vec{a}-\vec{c}$, $\vec{b}$ तथा $\vec{d}$ असमतलीय हैं।

माना कि बिन्दुओं P तथा Q के स्थिति सदिश क्रमशः $\vec{a}+\lambda_1\vec{b}$ तथा $\vec{a}+\lambda_2\vec{b}$ हैं।

माना कि बिन्दुओं R तथा S के स्थिति सदिश क्रमशः $\vec{c}+\mu_1\vec{d}$ तथा $\vec{c}+\mu_2\vec{d}$ हैं।

तो रेखाएँ PR तथा QS समान्तर हैं यदि और केवल यदि $\vec{c}-\vec{a}+\mu_1\vec{d}-\lambda_1\vec{b}=k\,(\vec{c}-\vec{a}+\mu_2\vec{d}-\lambda_2\vec{b})$

i.e. $(1-k)\,(\vec{c}-\vec{a}) + (\mu_1-k\mu_2)\vec{d} - (\lambda_1-k\lambda_2)\vec{b} = \vec{o}$

$\therefore$ $1-k=0,\ \mu_1-k\mu_2 = 0,\ \lambda_1-k\lambda_2=0$

i.e. $\mu_1=\mu_2$ तथा $\lambda_1=\lambda_2$ जो कि संभव नहीं है।

$\therefore$ PR, QS के समान्तर नहीं हो सकती है।

(ii) दी गई समीकरण $\dfrac{x-5}{3}=\dfrac{y-7}{-1}=\dfrac{z+2}{1}$ ... (i) तथा $\dfrac{x+3}{-3}=\dfrac{y-3}{2}=\dfrac{z-6}{4}$ ... (ii)

कोई बिन्दु P रेखा (i) पर $(3r_1+5, -r_1+7, r_1-2)$ तथा रेखा (ii) पर कोई बिन्दु Q $(-3r_2-3, 2r_2+3, 4r_2+6)$ है।

PQ के दिक्अनुपात निम्न है–

$(3r_1+3r_2+8, -r_1-2r_2+4, r_1-4r_2-8)$ ... (iii)

माना कि (iii) में दिये गये दिक्अनुपात रेखा जिसके दिक्अनुपात 2, 7, – 5 है , के समानुपाती होगें।

$$\therefore \quad \frac{3r_1+3r_2+8}{2}=\frac{-\ r_1-2r_2+4}{7}=\frac{r_1-4r_2-8}{-5} \quad \text{... (iv)}$$

(iv) को हल करने पर प्राप्त होता है कि $r_1=r_2=-1$

अतः प्रतिच्छेदन बिन्दु P(2, 8, – 3) तथा Q(0, 1, 2) हैं।

तथा अन्तःखण्डित लम्बाई $= PQ = \sqrt{(2-0)^2+(8-1)^2+(-\ 3-2)^2} = \sqrt{78}$

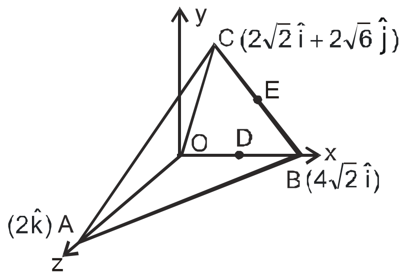
तथा PQ का समीकरण $\frac{x-2}{2}=\frac{y-8}{7}=\frac{z+3}{-5}$ है।

16. The base of the pyramid AOBC is an equilateral triangle OBC with each side equal to $4\sqrt{2}$. 'O' is the origin of reference, AO is perpendicular to the plane of Δ OBC and $|\overrightarrow{AO}| = 2$. Then find the cosine of the angle between the skew straight lines one passing through A and the mid point of OB and the other passing through 'O' and the mid point of BC.

एक पिरामिड AOBC का आधार एक समबाहु त्रिभुज OBC है, जिसकी प्रत्येक भुजा $4\sqrt{2}$ के बराबर है। 'O' को मूल बिन्दु माना गया है। AO, ΔOBC के समतल के लम्बवत् है और $|\overrightarrow{AO}| = 2$ है। दो विषमतलीय रेखाओं के बीच के कोण की कोज्या (cosine) ज्ञात कीजिए जबकि एक रेखा A और OB के मध्य बिन्दु से गुजरती है तथा दूसरी रेखा 'O' तथा BC के मध्य बिन्दु से गुजरती है।

Ans. $\frac{1}{\sqrt{2}}$

Sol.



$\overrightarrow{AD} = 2\sqrt{2}\,\hat{i} - 2\hat{k}$

$\overrightarrow{OE} = 3\sqrt{2}\,\hat{i} + \sqrt{6}\,\hat{j}$

$\cos\theta = \frac{12}{\sqrt{12}\sqrt{24}}$

$\cos\theta = \frac{1}{\sqrt{2}}$

17. If D, E, F be three point on BC, CA, AB respectively of a ΔABC. Such that the line AD, BE, CF are concurrent then find the value of $\frac{BD}{CD}\cdot\frac{CE}{AE}\cdot\frac{AF}{BF}$.

यदि ΔABC के BC, CA तथा AB पर क्रमशः तीन बिन्दु D, E, F इस प्रकार है कि रेखाऐं AD, BE, CF संगामी है तब $\frac{BD}{CD}\cdot\frac{CE}{AE}\cdot\frac{AF}{BF}$. का मान ज्ञात कीजिए –

Ans. 1

Sol. $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{p}$

$x + y + z + w = 0$

From the two equation दो समीकरणों से

$\frac{x\vec{a}+y\vec{b}}{x+y} = \frac{z\vec{c}+w\vec{p}}{z+w}$

Position vector of F का स्थिति सदिश $= \frac{x\vec{a}+y\vec{b}}{x+y}$

F divide AB in the ratio $\Rightarrow \frac{y}{x}$

F, AB को $\frac{y}{x}$ के अनुपात में विभाजित करता है।

$\frac{AF}{FB} = \frac{y}{x}$

Similarly इसी प्रकार $\frac{BD}{CD} = \frac{z}{y}$ & $\frac{CE}{AE} = \frac{x}{z}$; $\frac{AF}{BF} \cdot \frac{BD}{CD} \cdot \frac{CE}{AE} = 1$

18. Without expanding the determinant, Prove that

$$\begin{vmatrix} na_1+b_1 & na_2+b_2 & na_3+b_3 \\ nb_1+c_1 & nb_2+c_2 & nb_3+c_3 \\ nc_1+a_1 & nc_2+a_2 & nc_3+a_3 \end{vmatrix} = (n^3+1)\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

सारणिक का विस्तार किए बिना सिद्ध कीजिए

$$\begin{vmatrix} na_1+b_1 & na_2+b_2 & na_3+b_3 \\ nb_1+c_1 & nb_2+c_2 & nb_3+c_3 \\ nc_1+a_1 & nc_2+a_2 & nc_3+a_3 \end{vmatrix} = (n^3+1)\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Sol. Let's consider माना कि

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

we have to prove हमें सिद्ध करना है

$\left[n\vec{a}+\vec{b} \quad n\vec{a}+\vec{c} \quad n\vec{c}+\vec{a}\right] = (n^3+1)\left[\vec{a} \quad \vec{b} \quad \vec{c}\right]$

$\left(n\vec{a}+\vec{b}\right).\left(\left(n\vec{b} + \vec{c}\right)\times\left(n\vec{c}+\vec{a}\right)\right)$

$\Rightarrow \quad (n\vec{a}+\vec{b}).[n^2(\vec{b}\times\vec{c}) + n(\vec{b}\times\vec{a}) + n(\vec{c}\times\vec{c}) + \vec{c}\times\vec{a}] \qquad [\vec{c}\times\vec{c} = 0]$

$\Rightarrow \quad n\vec{a}.(n^2(\vec{b} \times\vec{c}) + n(\vec{b}\times\vec{a}) + \vec{c}\times\vec{a}] + \vec{b}.(n^2(\vec{b}\times\vec{c}) + n(\vec{b}\times\vec{a}) + (\vec{c}\times\vec{a}))$

$n^3[\vec{a} \quad \vec{b} \quad \vec{c}] + [\vec{b} \quad \vec{c} \quad \vec{a}]$

$(n^3 + 1)\ [\vec{a} \quad \vec{b} \quad \vec{c}]$

19. (i) OABC is a regular tetrahedron D is circumcentre of Δ OAB and E is mid point of edge AC. Prove that DE is equal to half the edge of tetrahedron.

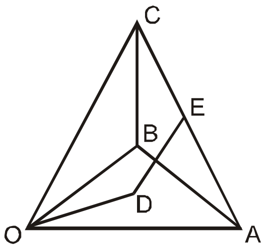
(ii) If V be the volume of a tetrahedron and V′ be the volume of the tetrahedron formed by the centroids and V = k V′ then find the value of k.

Ans. (ii) 27

(i) OABC एक समचतुष्फलक है, ΔOAB का परिकेन्द्र D है और किनारे (Edge) AC का मध्य बिन्दु E है। सिद्ध कीजिए कि DE, समचतुष्फलक की किनार (Edge) की आधी है।

(ii) यदि V समचतुष्फलक का आयतन है और V′ केन्द्रकों के द्वारा बनाये गये समचतुष्फलक का आयतन है तथा V = kV′. तो k का मान ज्ञात कीजिए ।

Sol. (i)



$\overrightarrow{OD} = \dfrac{\overrightarrow{OA}+\overrightarrow{OB}}{3} = \dfrac{\vec{a}+\vec{b}}{3}$ ($\because$ ΔOAB is an equilateral Δ)

$\overrightarrow{OE} = \dfrac{\vec{a}+\vec{c}}{2}$

$\therefore$ $\overrightarrow{DE} = \dfrac{\vec{a}+3\vec{c}-2\vec{b}}{6}$

$|\overrightarrow{DE}| = \sqrt{\left(\dfrac{\vec{a}+3\vec{c}-2\vec{b}}{6}\right)^2}$ ($\because |\vec{a}| = |\vec{b}| = |\vec{c}|$ and $\theta = 60°$)

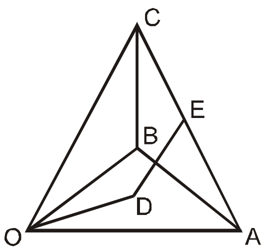
$\Rightarrow$ $DE = \dfrac{1}{6}\sqrt{a^2+9a^2+4a^2+\dfrac{6a^2}{2}-\dfrac{12a^2}{2}-\dfrac{4a^2}{2}}$ $\Rightarrow$ $DE = \dfrac{1}{6}\sqrt{9a^2}$ $\Rightarrow$ $DE = \dfrac{a}{2}$

(ii) $V = \dfrac{1}{6}[\vec{a}\ \ \vec{b}\ \ \vec{c}]$

The centroid are $\dfrac{\vec{a}+\vec{b}}{3}, \dfrac{\vec{b}+\vec{c}}{3}, \dfrac{\vec{c}+\vec{a}}{3}, \dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$

$\therefore$ $V' = \dfrac{1}{6}\begin{bmatrix}\dfrac{\vec{c}-\vec{a}}{3} & \dfrac{\vec{c}-\vec{b}}{3} & \dfrac{\vec{c}}{3}\end{bmatrix} = \dfrac{1}{6\times 27}[\vec{a}\ \ \vec{b}\ \ \vec{c}] = \dfrac{1}{27}V$ $\therefore$ k = 27

Hindi (i)



$\overrightarrow{OD} = \dfrac{\overrightarrow{OA}+\overrightarrow{OB}}{3} = \dfrac{\vec{a}+\vec{b}}{3}$ ($\because$ ΔOAB एक समबाहु त्रिभुज है।)

$\overrightarrow{OE} = \dfrac{\vec{a}+\vec{c}}{2}$

$\therefore$ $\overrightarrow{DE} = \dfrac{\vec{a}+3\vec{c}-2\vec{b}}{6}$

$|\overrightarrow{DE}| = \sqrt{\left(\dfrac{\vec{a}+3\vec{c}-2\vec{b}}{6}\right)^2}$ ($\because |\vec{a}| = |\vec{b}| = |\vec{c}|$ तथा $\theta = 60°$)

$\Rightarrow$ $DE = \dfrac{1}{6}\sqrt{a^2+9a^2+4a^2+\dfrac{6a^2}{2}-\dfrac{12a^2}{2}-\dfrac{4a^2}{2}}$ $\Rightarrow$ $DE = \dfrac{1}{6}\sqrt{9a^2}$ $\Rightarrow$ $DE = \dfrac{a}{2}$

(ii) $V = \dfrac{1}{6}[\vec{a}\ \ \vec{b}\ \ \vec{c}]$

केन्द्रक $\dfrac{\vec{a}+\vec{b}}{3}, \dfrac{\vec{b}+\vec{c}}{3}, \dfrac{\vec{c}+\vec{a}}{3}, \dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$ है।

$\therefore \qquad V' = \dfrac{1}{6}\begin{bmatrix} \dfrac{\vec{c}-\vec{a}}{3} & \dfrac{\vec{c}-\vec{b}}{3} & \dfrac{\vec{c}}{3} \end{bmatrix} = \dfrac{1}{6\times 27}[\vec{a} \;\; \vec{b} \;\; \vec{c}] = \dfrac{1}{27}V \qquad \therefore \qquad k = 27$

20. Given that $\vec{u} = \hat{i} - 2\hat{j} + 3\hat{k}$ $\vec{v} = 2\hat{i} + \hat{j} + 4\hat{k}$, $\vec{w} = 2\hat{i} + j + 3\hat{k}$ and $(\vec{u}.\vec{R} - 10)\, i + (\vec{v}.\vec{R} - 20)\, j + (\vec{w}.\vec{R} - 20)\, \hat{k} = \vec{0}$ then find $\vec{R}$

Ans. $\vec{R} = 10\hat{i}$

दिया गया है $\vec{u} = \hat{i} - 2\hat{j} + 3\hat{k}$ $\vec{v} = 2\hat{i} + \hat{j} + 4\hat{k}$, $\vec{w} = 2\hat{i} + j + 3\hat{k}$ और $(\vec{u}.\vec{R} - 10)\, i + (\vec{v}.\vec{R} - 20)\, j + (\vec{w}.\vec{R} - 20)\, \hat{k} = \vec{0}$ तब $\vec{R}$ को ज्ञात कीजिए।

Sol. $(\vec{u}.\vec{R})\hat{i} + (\vec{v}.\vec{R})\hat{j} + (\vec{w}.\vec{R})\hat{k} = 10\hat{i} + 20\hat{j} + 20\hat{k}$

$\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{u}.\vec{R} = x - 2y + 3z$

$\vec{v}.\vec{R} = 2x + y + 4z$

$\vec{w}.\vec{R} = x + 3y + 3z$

$(x - 2y + 3z)\ \hat{i} + (2x + y + 4z)\ \hat{j} + (x + 3y + 3z)\ \hat{k} = 10\hat{i} + 20\hat{j} + 20\hat{k}$

$\Rightarrow \qquad x - 2y + 3z = 10 \qquad$ ....(1)

$2x + y + 4z = 20 \qquad$ ....(2)

$2x + y + 3z = 20 \qquad$ ....(3)

$(x, y, z) = (10, 2, 5)$

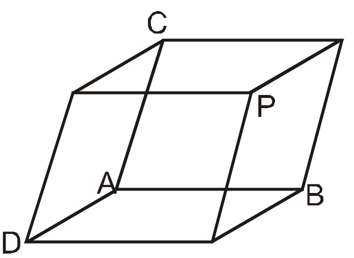
$\vec{R} = 10\hat{i}$

21. $\overline{AB}$, $\overline{AC}$ and $\overline{AD}$ are three adjacent edges of a parallelopiped . The diagonal of the parallelopiped passing through A and directed away from it is vector $\vec{a}$. The vector area of the faces containing vertices A , B , C and A , B , D are $\vec{b}$ and $\vec{c}$ respectively i.e. $\overrightarrow{AB} \times \overrightarrow{AC} = \vec{b}$ and $\overrightarrow{AD} \times \overrightarrow{AB} = \vec{c}$. If projection of each edge AB and AC on diagonal vector $\vec{a}$ is $\dfrac{|\vec{a}|}{3}$, then find the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ in terms of $\vec{a}$, $\vec{b}$, $\vec{c}$ and $|\vec{a}|$.

$\overline{AB}$, $\overline{AC}$ और $\overline{AD}$ एक घनाभ की तीन आसन्न भुजायें है। घनाभ के शीर्ष बिन्दु A से गुजरने वाला और इससे दूर की ओर जाती दिशा में विकर्ण सदिश $\vec{a}$ है। फलकें जिनके शीर्ष A , B , C तथा A , B , D हैं के क्षेत्रफल क्रमशः सदिश $\vec{b}$ तथा $\vec{c}$ है। अर्थात् $\overrightarrow{AB} \times \overrightarrow{AC} = \vec{b}$ और $\overrightarrow{AD} \times \overrightarrow{AB} = \vec{c}$ यदि भुजा AB और AC प्रत्येक का, विकर्ण $\vec{a}$ पर प्रक्षेप $\dfrac{|\vec{a}|}{3}$ है, तो $\vec{a}$, $\vec{b}$, $\vec{c}$ तथा $|\vec{a}|$ के पदों में सदिश $\vec{AB}$, $\vec{AC}$ तथा $\vec{AD}$ को ज्ञात कीजिए।

Ans. $\overrightarrow{AB} - \overrightarrow{AD} = 3\dfrac{\vec{c}\times\vec{a}}{|\vec{a}|^2}$; $\overrightarrow{AC} = \dfrac{1}{3}\vec{a} + \dfrac{\vec{a}\times(\vec{b}-\vec{c})}{|\vec{a}|^2} + \dfrac{3(\vec{b}\times\vec{a})}{|\vec{a}|^2}$; $\overrightarrow{AD} = \dfrac{1}{3}\vec{a} + \dfrac{\vec{a}\times(\vec{b}-\vec{c})}{|\vec{a}|^2} - \dfrac{3(\vec{c}\times\vec{a})}{|\vec{a}|^2}$

Sol.



$\vec{a} = \overrightarrow{AP} = \overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} \qquad$ ......(i)

$\overrightarrow{AB} \times \overrightarrow{AC} = \vec{b}$

$\overrightarrow{AD} \times \overrightarrow{AB} = \vec{c}$

$\overrightarrow{AB}.\dfrac{\vec{a}}{|\vec{a}|} = \dfrac{|\vec{a}|}{3}$ i.e. $\overrightarrow{AB}.\vec{a} = \dfrac{|\vec{a}\ |^2}{3}$

$\overrightarrow{AC}.\dfrac{\vec{a}}{|\vec{a}|} = \dfrac{|\vec{a}|}{3}$ i.e. $\overrightarrow{AC}\ .\ \vec{a} = \dfrac{|\vec{a}\ |^2}{3}$

$\therefore \quad (\overrightarrow{AB} \times \overrightarrow{AC}) \times \vec{a} = \vec{b} \times \vec{a}$

$\therefore \quad \dfrac{|\vec{a}|^2}{3}\ \overrightarrow{AC} - \dfrac{|\vec{a}|^2}{3}\overrightarrow{AB} = \vec{b} \times \vec{a}$

$\therefore \quad \overrightarrow{AC} - \overrightarrow{AB} = 3\dfrac{\vec{b} \times \vec{a}}{|\vec{a}|^2}$ ....(ii)

$|\vec{a}|^2 = \overrightarrow{AB}\ .\ \vec{a} + \overrightarrow{AC}\ .\ \vec{a} + \overrightarrow{AD}\ .\ \vec{a}$

$\therefore \quad \dfrac{|\vec{a}|^2}{3} = \overrightarrow{AD}.\vec{a}$

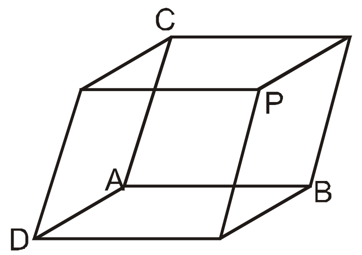
$(\overrightarrow{AD} \times \overrightarrow{AB}) \times \vec{a} = \vec{c} \times \vec{a} \quad \therefore \quad \overrightarrow{AB} - \overrightarrow{AD} = 3\dfrac{\vec{c} \times \vec{a}}{|\vec{a}|^2}$ ....(iii)

from (i) , (ii) and (iii), we get $\overrightarrow{AB} = \dfrac{1}{3}\vec{a} + \dfrac{\vec{a} \times (\vec{b} - \vec{c})}{|\vec{a}|^2}$

Now from (ii) and (iii), we can easily find $\overrightarrow{AC}$ and $\overrightarrow{AD}$

$\overrightarrow{AC} = \dfrac{1}{3}\vec{a} + \dfrac{\vec{a} \times (\vec{b} - \vec{c})}{|\vec{a}|^2} + \dfrac{3\ \ (\vec{b} \times \vec{a})}{|\vec{a}|^2}$; $\overrightarrow{AD} = \dfrac{1}{3}\vec{a} + \dfrac{\vec{a} \times (\vec{b} - \vec{c})}{|\vec{a}|^2} - \dfrac{3\ \ (\vec{c} \times \vec{a})}{|\vec{a}|^2}$

Hindi



$\vec{a} = \overrightarrow{AP} = \overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD}$ ......(i)

$\overrightarrow{AB} \times \overrightarrow{AC} = \vec{b}$

$\overrightarrow{AD} \times \overrightarrow{AB} = \vec{c}$

$\overrightarrow{AB}.\dfrac{\vec{a}}{|\vec{a}|} = \dfrac{|\vec{a}|}{3}$ अर्थात् $\overrightarrow{AB}.\vec{a} = \dfrac{|\vec{a}\ |^2}{3}$

$\overrightarrow{AC}.\dfrac{\vec{a}}{|\vec{a}|} = \dfrac{|\vec{a}|}{3}$ अर्थात् $\overrightarrow{AC}\ .\ \vec{a} = \dfrac{|\vec{a}\ |^2}{3}$

$\therefore \quad (\overrightarrow{AB} \times \overrightarrow{AC}) \times \vec{a} = \vec{b} \times \vec{a}$

$\therefore \quad \dfrac{|\vec{a}|^2}{3}\ \overrightarrow{AC} - \dfrac{|\vec{a}|^2}{3}\overrightarrow{AB} = \vec{b} \times \vec{a}$

$\therefore \quad \overrightarrow{AC} - \overrightarrow{AB} = 3\dfrac{\vec{b} \times \vec{a}}{|\vec{a}|^2}$ ....(ii)

$|\vec{a}|^2 = \overrightarrow{AB}\ .\ \vec{a} + \overrightarrow{AC}\ .\ \vec{a} + \overrightarrow{AD}\ .\ \vec{a}$

$\therefore \quad \dfrac{|\vec{a}|^2}{3} = \overrightarrow{AD}.\vec{a}$

$(\overrightarrow{AD} \times \overrightarrow{AB}) \times \vec{a} = \vec{c} \times \vec{a} \quad \therefore \quad \overrightarrow{AB} - \overrightarrow{AD} = 3\dfrac{\vec{c}\times\vec{a}}{|\vec{a}|^2}$ ....(iii)

समीकरण (i) , (ii) और (iii) से $\quad \overrightarrow{AB} = \dfrac{1}{3}\vec{a} + \dfrac{\vec{a}\times(\vec{b}-\vec{c})}{|\vec{a}|^2}$

समीकरण (ii) और (iii) से $\overrightarrow{AC}$ and $\overrightarrow{AD}$

$\overrightarrow{AC} = \dfrac{1}{3}\vec{a} + \dfrac{\vec{a}\times(\vec{b}-\vec{c})}{|\vec{a}|^2} + \dfrac{3 \quad (\vec{b}\times\vec{a})}{|\vec{a}|^2} \quad ; \quad \overrightarrow{AD} = \dfrac{1}{3}\vec{a} + \dfrac{\vec{a}\times(\vec{b}-\vec{c})}{|\vec{a}|^2} - \dfrac{3 \quad (\vec{c}\times\vec{a})}{|\vec{a}|^2}$

22. Prove that if the equation $\vec{r}\times\vec{a} = \vec{b}$ and $\vec{r}\times\vec{c} = \vec{d}$ are consistent $\left(\vec{a}.\vec{d} \neq 0, \vec{b}.\vec{c} \neq 0, \vec{d}\times\vec{b} \neq \vec{0}\right)$ then $\vec{b}.\vec{c} + \vec{a}.\vec{d} = 0$

सिद्ध कीजिए समीकरण $\vec{r}\times\vec{a} = \vec{b}$ और $\vec{r}\times\vec{c} = \vec{d}$ के निकाय संगत है $\left(\vec{a}.\vec{d} \neq 0, \vec{b}.\vec{c} \neq 0, \vec{d}\times\vec{b} \neq \vec{0}\right)$ तब $\vec{b}.\vec{c} + \vec{a}.\vec{d} = 0$

Sol. $\vec{r}\times\vec{a} = \vec{b} \Rightarrow \vec{d}\times(\vec{r}\times\vec{a}) = \vec{d}\times\vec{b}$

$\Rightarrow \left(\vec{a}.\vec{d}\right)\vec{r} - \left(\vec{d}.\vec{r}\right)\vec{a} = \vec{d}\times\vec{b} \Rightarrow \vec{r} = \dfrac{\vec{d}\times\vec{b}}{\vec{a}.\vec{d}}$ ......(i)

Now अब $\vec{r}\times\vec{c} = \vec{d} \Rightarrow \vec{b}\times(\vec{r}\times\vec{c}) = \vec{b}\times\vec{d}$

$\Rightarrow \left(\vec{b}.\vec{c}\right)\vec{r} - \left(\vec{b}.\vec{r}\right)\vec{c} = \vec{b}\times\vec{d} \Rightarrow \quad \vec{r} = \dfrac{\vec{d}\times\vec{b}}{-\vec{b}.\vec{c}}$ .....(ii)

from (i) and (ii) we get $\vec{a}.\vec{d} = -\vec{b}.\vec{c}$ ((i) और (ii) से $\vec{a}.\vec{d} = -\vec{b}.\vec{c}$)